Finite Mathematics & Modeling

The Community College of Baltimore County
MATH 125 Textbook: 2018 Edition

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Table of Contents

Chapter 1: Statistics

Section 1.1: Terminology

World View Note: The term “statistics” emerged around the 18th century due to the need of governments to collect demographic and economic data. Nowadays, some of the fields that use statistics include finance, actuarial science, and biostatistics.

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950’s and 60’s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

Populations and Samples

Before we begin gathering and analyzing data, we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of Washington.
- All first-year students at public colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Population
The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Washington might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from the University of Washington. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study.

Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 2

A researcher wanted to know how citizens of Tacoma felt about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

Try it Now 1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Categorizing Data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.

Quantitative and Categorical Data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories. **Quantitative data** are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater. What would be an example of categorical data?

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 4

What kind of survey could we conduct with involving movies that yields quantitative data?
A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...)

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location. What kind of data would this be?

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1 - it was awful
2 - it was just OK
3 - I liked it
4 - it was great
5 - best movie ever!

How can we organize the results?

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer...well, *Finding Nemo*.

Try it Now 3

Classify each measurement as categorical or quantitative
a. Eye color of a group of people
b. Daily high temperature of a city over several weeks
c. Annual income

Try it Now Answers

1. The population is trout in the lake, and the sample is the 20 fish that were caught and measured.

2. This would be a parameter, as there is no indication that the average age came from a smaller sample. Since it came from the whole college (population), this is a parameter.

3. Quantitative = annual income and daily high temperature
Qualitative/Categorical = eye color of a group of people

Section 1.1 Exercises

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said they were not supporting the bill, and 2 were undecided.

    a. What is the population of this survey?
    b. What is the size of the population?
    c. What is the size of the sample?
    d. Give the sample statistic for the proportion of voters surveyed who said they were supporting the education bill.
    e. Based on this sample, we might expect how many of the representatives to support the education bill?

2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they would vote for Brown, 207 said they would vote for Feliz, and 31 were undecided.

    a. What is the population of this survey?
    b. What is the size of the population?
    c. What is the size of the sample?
    d. Give the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
    e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?

3. Suppose researchers decided to compare the box office returns for the past year of action, drama, and horror movies.

    a. Which of the above would be categorical data?
    b. Which information about would be quantitative?

4. A survey of the population of moviegoers in a city talks to 300 patrons. They are asked if they prefer Action, Comedy, or Drama.

    a. What is the sample size?
    b. Are the choices categorical or quantitative?
    c. Suppose 125 said they prefer Action movies. Based on this statistic, how many from a group of 1200 people could we expect to see Action movies?

Section 1.1 Exercises – Answer Key

1. a. representatives in a state's congress
   b. 106
   c. 28
   d. 14/28 were supporting the education bill, or half
   e. half of 106, or 53

2. a. registered voters in Raleigh
   b. 9500
   c. 350
   d. 112/350 voted Brown, or 0.32
   e. 3040

3. a. action, drama, and horror movie genres
   b. the box office returns

4. a. 300 patrons
   b. categorical
   c. 500

Section 1.2: Describing Data

World View Note: John Graunt (1620-1674) was the first person to draw an extensive set of statistical inferences from mass data. He launched the discipline of mathematical statistics. Also, he was the only shopkeeper in the original 119 fellows in the Royal School of London.

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data then explore numerical summaries of data.

Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Frequency Table
A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. They find that 52 accidents involve green cars, 41 with red, 39 with black, 36 with white, 23 with grey, and 25 with blue. Create a frequency chart.

Grouping the data we have the following summary.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 2

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

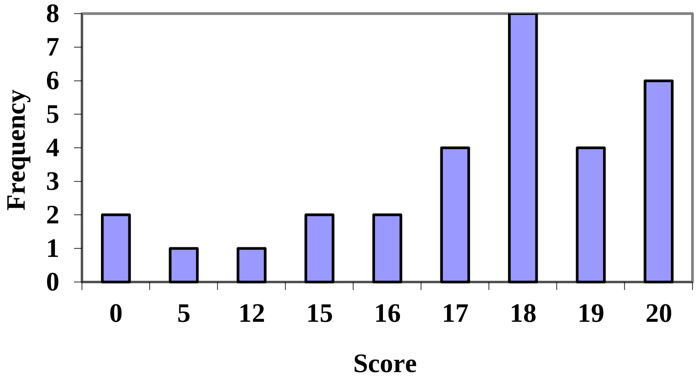
19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

Create a frequency table and bar graph.

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary.

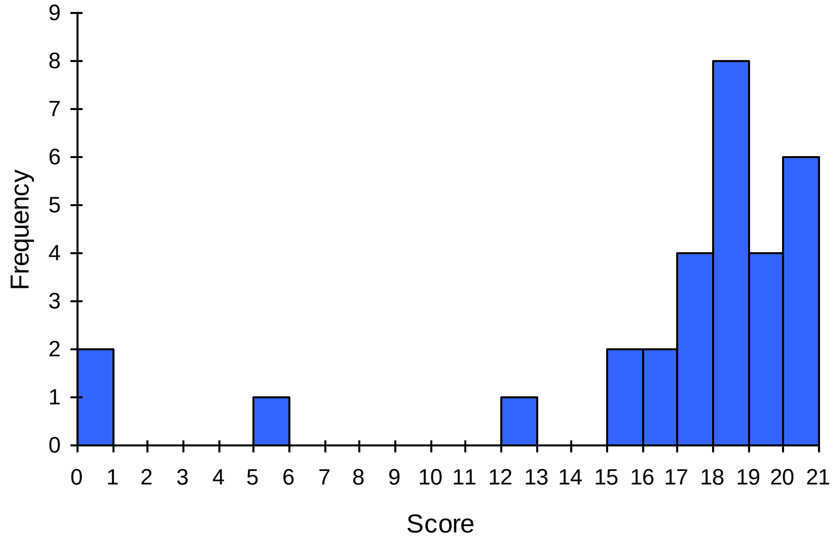


However, since the scores are numerical values, this chart doesn’t really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram
A **histogram** is like a bar graph, but where the horizontal axis is a number line

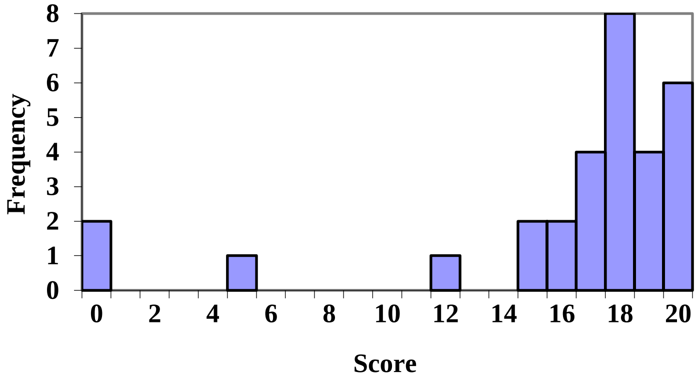
Example 3

For the values above, create a histogram.



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.



If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

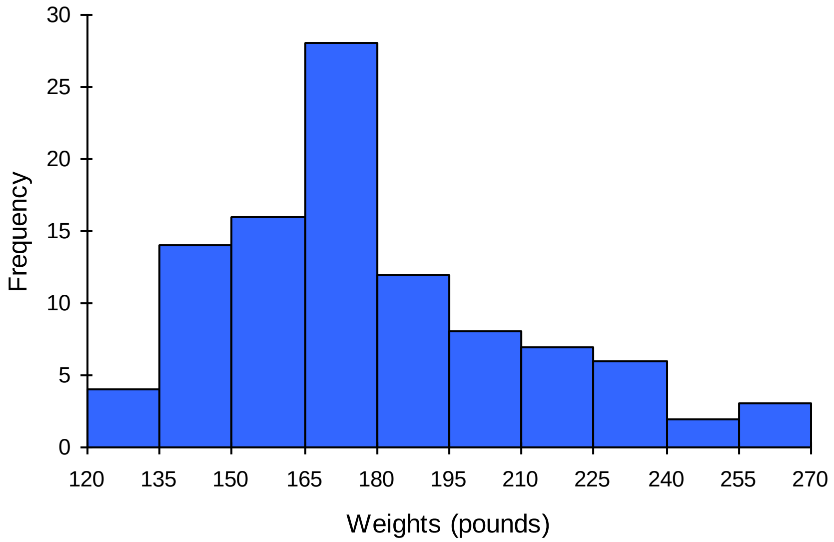
- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 4

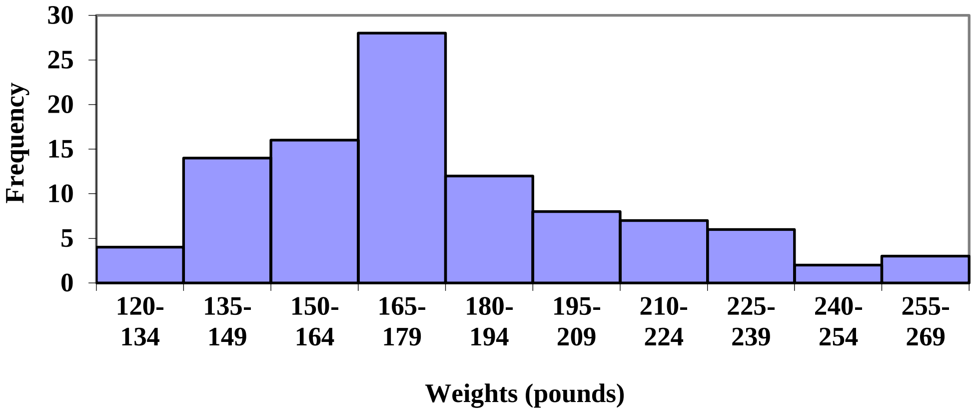
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	Frequency
120 - 134	4
135 – 149	14
150 – 164	16
165 – 179	28
180 – 194	12
195 – 209	8
210 – 224	7
225 – 239	6
240 – 254	2
255 - 269	3

Create a histogram of the data:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Try it Now 1

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

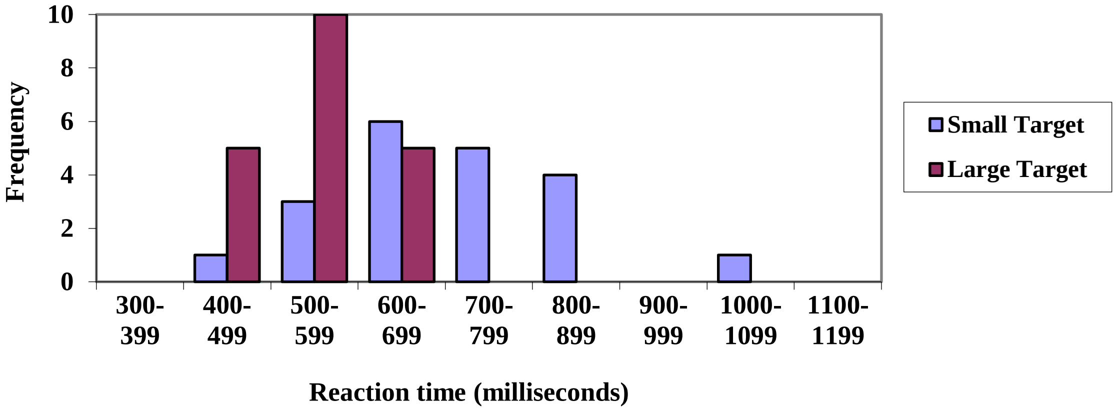
Example 5

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	Frequency (small target)	Frequency (large target)
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0
1100-1199	0	0

Create a histogram that compares the frequencies of the small and large targets.

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.



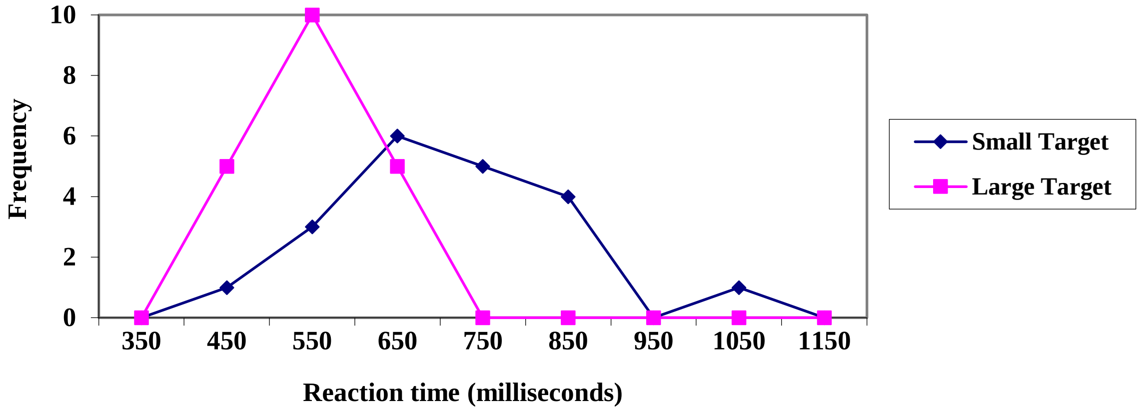
Frequency Polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 6

Create frequency polygon for the large and small target data.

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



Try it Now Answers

1. Using a class intervals of size 55, we can group our data into six intervals and use the frequency distribution to generate the histogram.

Cost interval	Frequency
$140-194	5
$195-249	3
$250-304	9
$305-359	12
$360-414	4
$415-469	3

Section 1.2 Exercises

1. The table below shows scores on a math test.

    a. Complete the frequency table for the math test scores
    b. Construct a histogram of the data

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

2. A group of adults where asked how many cars they had in their household

    a. Complete the frequency table for the car number data
    b. Construct a histogram of the data

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

3. Twenty-four students were asked the number of hours they sleep each night. The results of the survey are below

    a. Complete the frequency table for the number of hours of sleep each student got each night. (5 intervals with a class width of 2)
    b. Construct a histogram of the data.

7	8	10	8	9	12	6	10
5	9	11	7	5	12	8	4
7	6	3	6	11	8	7	9

4. Fifteen AA batteries were tested to determine how long, in hours, they would last. The results are below.

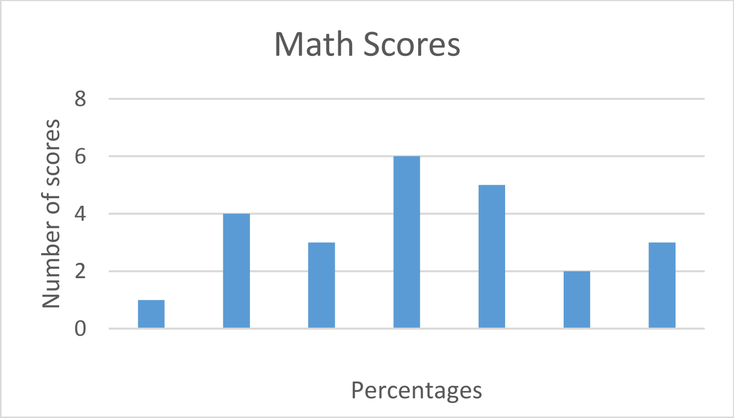
    a. Compute the frequency table for the fifteen AA batteries that were tested (using 5 intervals with class width of 6).
    b. Construct a histogram of the data.

387	393	394	370	377
389	392	391	382	381
399	396	372	386	390

Section 1.2 Exercises – Answer Key

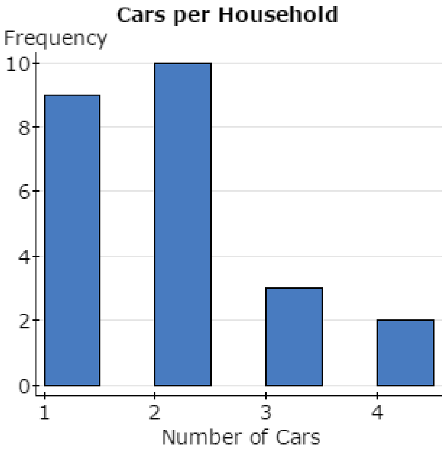
1.

Interval	**Frequency**
30-40	1
41-50	4
51-60	3
61-70	6
71-80	5
81-90	2
91-100	3



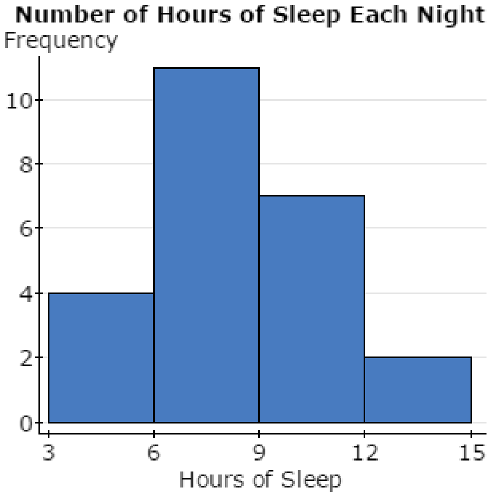
2.

Intervals	Frequency
1	9
2	10
3	3
4	2



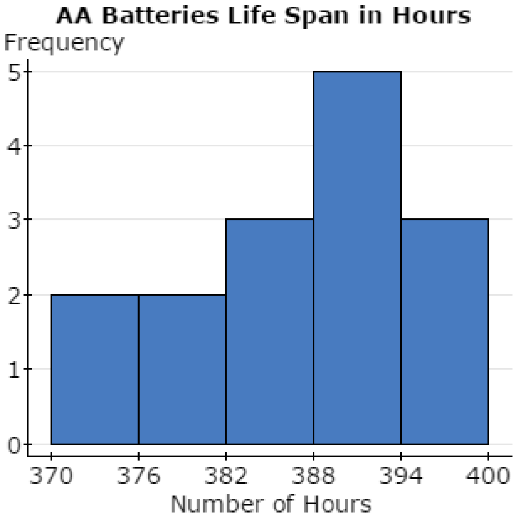
3.

Interval	Frequency
3 – 5	4
6 – 8	11
9 – 11	7
12 – 15	2



4.

Intervals	Frequency
370 – 375	2
376 – 381	2
382 – 387	3
388 – 393	5
394 – 399	3



Section 1.3: Measures of Central Tendency

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Let's begin by trying to find the most "typical" value of a data set. Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 1

Marci’s exam scores for her last math class were: 79, 86, 82, 94. What is the average (mean) of her scores?

The mean of these values is $\frac{79+86+82+94}{4}=85.25$.

Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 2

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Find the average number of touchdown passes thrown.

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get 634/31 = 20.4516. It would be appropriate to round this to 20.5.

It would be most correct for us to report that “The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes,” but it is not uncommon to see the more casual word “average” used in place of “mean.”

Try it Now 1

The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Example 3

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below. Find the mean.

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15+\cdots+15}^{\text{6 terms}}+\overbrace{20+\cdots+20}^{\text{8 terms}}+\overbrace{25+\cdots+25}^{\text{11 terms}}+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15\cdot 6=90$. Using this simplification, we get:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 4

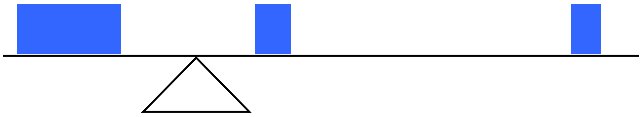
Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5,000 thousand). Recalculate the mean.

Adding this to our sample, our mean is now:

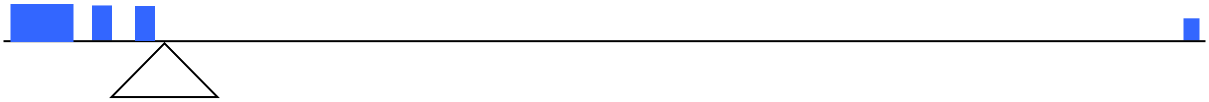
$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7+5000\cdot 1}{101}=\frac{8390}{101}=83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order.

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, *N*, is odd, then the median is the middle data value. This value can be found by rounding *N*/2 up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values *N*/2 and *N*/2 + 1)

Example 5

Find the median value for the football data from example 2.

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (31/2 = 15.5, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 6

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get (6+7)/2 = 6.5.

The median quiz score was 6.5.

Try it Now 2

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 7

Let us return now to our original household income data

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Here we have 100 data values. Find the median value.

If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so	Values 1 to 6 are $15 thousand
The next 8 data values are $20, so	Values 7 to (6+8)=14 are $20 thousand
The next 11 data values are $25, so	Values 15 to (14+11)=25 are $25 thousand
The next 17 data values are $30, so	Values 26 to (25+17)=42 are $30 thousand
The next 19 data values are $35, so	Values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 8

Refer again to the household income data. If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. Find the median.

As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 9

In our vehicle color survey, we collected the data below. Identify the mode.

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try it Now 3

Reviewers were asked to rate a product on a scale of 1 to 5. Determine:

a. The mean rating
b. The median rating
c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Try it Now Answers

1. Adding the prices and dividing by 5 we get the mean price: \$3.682

2. First we put the data in order: \$3.29, \$3.59, \$3.75, \$3.79, \$3.99. Since there are an odd number of data, the median will be the middle value, \$3.75.

3. There are 23 ratings.
   a. The mean is $\frac{1\cdot 4+2\cdot 8+3\cdot 7+4\cdot 3+5\cdot 1}{23}\approx 2.5$
   b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
   c. The mode is the most frequent rating. The mode rating is 2.

Section 1.3 Exercises

1. Determine the mean, median, and mode of weights of recent patients seen at a clinic: 126, 180, 217, 189, 154, 193, 190, 293, 315, 210, 184, 145, 173, 189

2. Determine the mean, median, and mode of the number of classes that students in a particular group are taking this semester: 2, 4, 5, 1, 1, 1, 3, 5, 6, 2, 3, 1, 3

3. Determine the mean, median, and mode of the number of daily calls experienced at a call center during the previous week: 315, 125, 140, 220, 154, 256, 492

4. Fifty individuals were surveyed and asked to identify the number of children they had living in their home. The responses are summarized in the following table. Determine the mean, median, and mode of the data.

Number of Children	Frequency
0	21
1	6
2	12
3	9
5	2

5. Twenty individuals were surveyed and asked to identify the number of times they have undergone an intensive medical surgery. The responses are summarized in the following table. Determine the mean, median, and mode of the data.

Number of Surgeries	Frequency
0	10
1	5
2	4
3	1

Section 1.3 Exercises – Answer Key

1. mean: 197, median: 189; mode: 189
2. mean: 2.8 median: 3 mode: 1
3. mean: 243.1; median: 220; mode: none
4. mean: 1.34 median: 1 mode: 0
5. mean: 0.8; median: 0.5 ; mode: 0

Section 1.4: Measures of Variation

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5 5
Section B: 0 0 0 0 0 10 10 10 10 10
Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B, half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Range
The range is the difference between the maximum value and the minimum value of the data set.

Example 1

Using the quiz scores from above,

For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0
For section B, the range is 10 since 10 – 0 = 10
For section C, the range is 2 since 6 – 4 = 2

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard Deviation
The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value - mean
0	0-5 = -5
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
10	10-5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value - mean	deviation squared
0	0-5 = -5	$(-5)^2 = 25$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
10	10-5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, *n*, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by *n* - 1 (in this case, 10 - 1 = 9).[1]

---

[1] The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{population standard deviation} = \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$$

$$\text{sample standard deviation} = \sqrt{\frac{50}{9}} \approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by *n*, the number of data values, if the data represents a whole population; divide by *n* – 1 if the data is from a sample.
5. Compute the square root of the result.

Example 2

Find the standard deviation for Section B from example 1.

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value - mean	deviation squared
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}}=\sqrt{\frac{250}{10}}=5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try it Now 1

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q_1) is the value so that 25% of the data values are below it; the third quartile (Q_3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:
Minimum, Q_1, Median, Q_3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q_1

Begin by ordering the data from smallest to largest

Compute the locator: $L = 0.25n$

If L is a decimal value:

- Round up to $L+$
- Use the data value in the $L+^{\text{th}}$ position

If L is a whole number:

- Find the mean of the data values in the L^{th} and $L+1^{\text{th}}$ positions.

To find the third quartile, Q_3

Use the same procedure as for Q_1, but with locator: $L = 0.75n$

Examples should help make this clearer.

Example 3

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are as listed. Find the first and third quartile.

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is $0.75(9) = 6.75$. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 4

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are below. Find the first and third quartiles.

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: $(60+62)/2 = 61$, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: (67+69)/2 = 68, so Q_3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 5

1. For the 9 female sample, find the five-number summary.

   The median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

2. For the 8 female sample, find the five-number summary.

   The median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary is: 59, 61, 65, 68, 70.

Example 6

Returning to our quiz score data. Find the five-number summaries for Sections A-D.

In each case, the first quartile locator is 0.25(10) = 2.5, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try it Now 2

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285
$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320
$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Example 7

Returning to the household income data from Section 1.3, create the five-number summary.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was \$35 thousand. We can see in the table that the minimum income is \$15 thousand, and the maximum is \$50 thousand.

To find Q_1, we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q_1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of \$15, so	Values 1 to 6 are \$15 thousand
The next 8 data values are \$20, so	Values 7 to (6+8)=14 are \$20 thousand
The next 11 data values are \$25, so	Values 15 to (14+11)=25 are \$25 thousand
The next 17 data values are \$30, so	Values 26 to (25+17)=42 are \$30 thousand

The 25th data value is \$25 thousand, and the 26th data value is \$30 thousand, so Q_1 will be the mean of these: $(25 + 30)/2 =$ \$27.5 thousand.

To find Q_3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q_3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are \$35, so	Values 43 to (42+19)=61 are \$35 thousand
The next 20 data values are \$40, so	Values 61 to (61+20)=81 are \$40 thousand

Both the 75th and 76th data values lie in this group, so Q_3 will be \$40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between \$40 thousand and \$50 thousand.

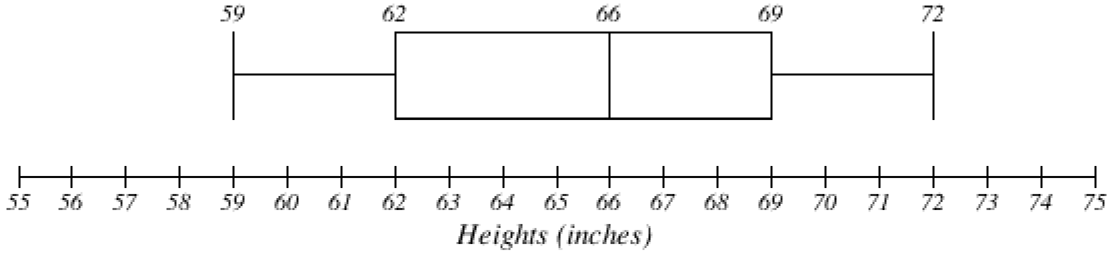
For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Box plot

A **box plot** is a graphical representation of a five-number summary. To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. "Whiskers" are extended out to the minimum and maximum values.

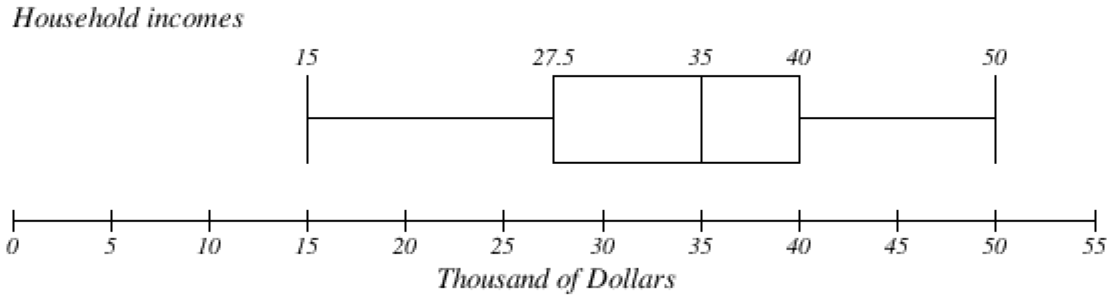
Example 8

Create a box plot based on the 9 female height data with 5 number summary: 59, 62, 66, 69, 72.



Example 9

Create a box plot based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50.



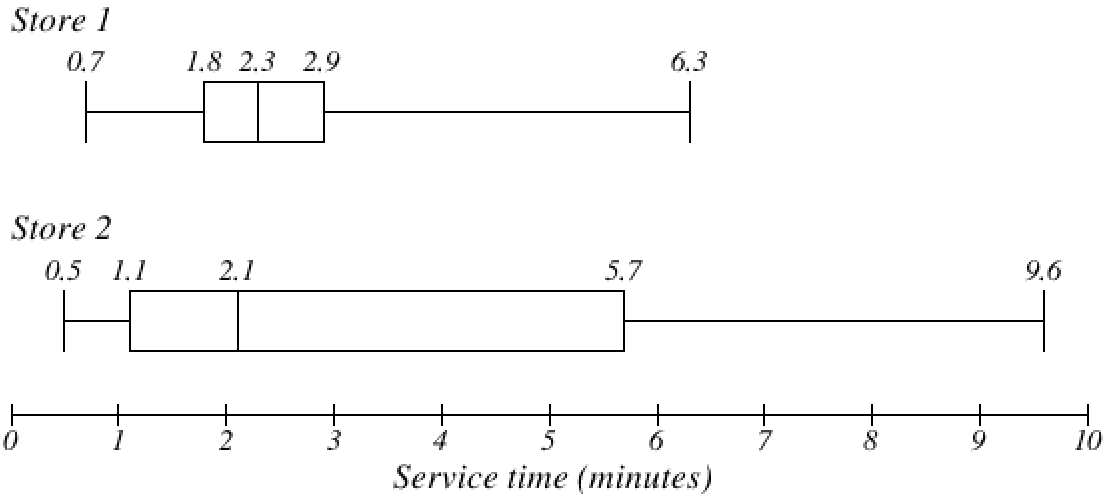
Try it Now 3

Create a boxplot based on the textbook price data from Try it Now 2.

Box plots are particularly useful for comparing data from two populations.

Example 10

The box plot of service times for two fast-food restaurants is shown below. Compare the two service times. Which is the best restaurant to go to?



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 11

The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS)[2]. Compare the two boxplots side by side.

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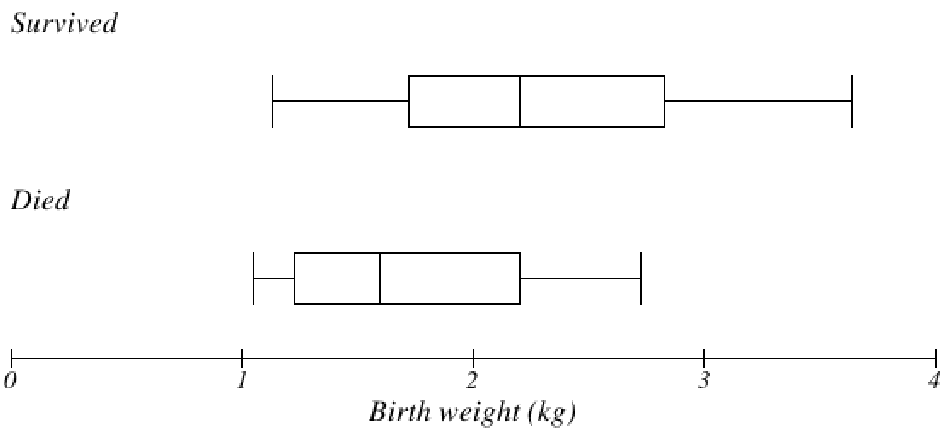
[2] van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, **48**, 249–255. As quoted on http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296&section=1.1.3



The boxplot is separated to show the birth weights of infants who survived and those that did not. Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

Try it Now Answers

1. Earlier we found the mean of the data was $3.682.

data value	deviation: data value - mean	deviation squared
3.29	3.29 – 3.682 = -0.391	0.153664
3.59	3.59 – 3.682 = -0.092	0.008464
3.79	3.79 – 3.682 = 0.108	0.011664
3.75	3.75 – 3.682 = 0.068	0.004624
3.99	3.99 – 3.682 = 0.308	0.094864

This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root:

$$\sqrt{\frac{0.153664+0.008464+0.011664+0.004624+0.094864}{4}} \approx \$0.261$$

2. The data is already in order, so we don't need to sort it first.
The minimum value is \$140 and the maximum is \$460.

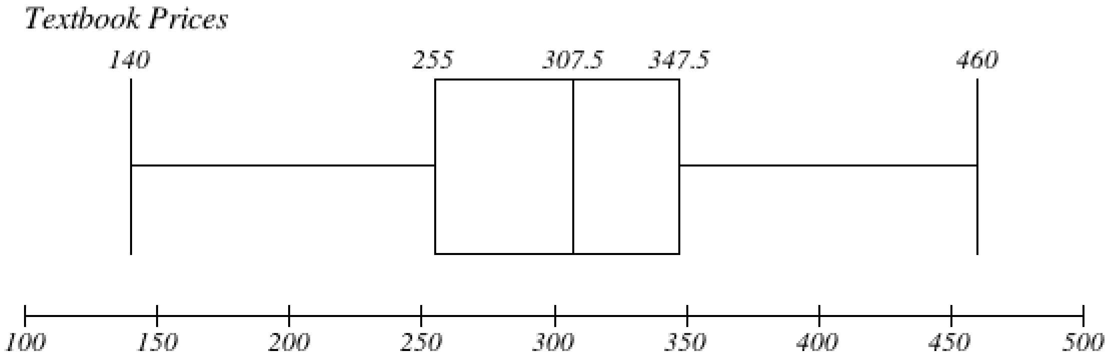
There are 36 data values so $n = 36$. $n/2 = 18$, which is a whole number, so the median is the mean of the 18th and 19th data values, \$305 and \$310. The median is \$307.50.

To find the first quartile, we calculate the locator, L = 0.25(36) = 9. Since this is a whole number, we know Q_1 is the mean of the 9th and 10th data values, \$250 and \$260. Q_1 = \$255.

To find the third quartile, we calculate the locator, L = 0.75(36) = 27. Since this is a whole number, we know Q_3 is the mean of the 27th and 28th data values, \$345 and \$350. Q_3 = \$347.50.

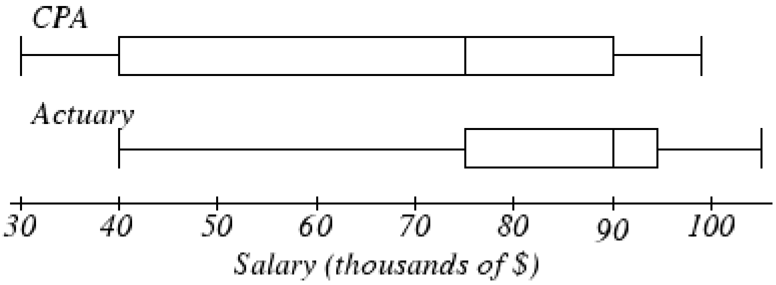
The 5 number summary of this data is: \$140, \$255, \$307.50, \$347.50, \$460

5.



Section 1.4 Exercises

1. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



2. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

3. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected. Create comparative boxplots for the data.

   Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

   Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

4. Create a boxplot for the data: 1, 2, 3, 3, 4, 5, 7, 10, 12, 15, 23, 34, 51, 52, 61, 79, 100, 127, 147, 260, 345

5. Determine the standard deviation for the **sample** data: 5, 3, 7, 14, 16

6. Determine the standard deviation for the **population** data: 14, 19, 34, 41

7. Make up three data sets with 5 numbers each that have:
   a. the same mean but different standard deviations.
   b. the same mean but different medians.
   c. the same median but different means.

8. Consider the salaries listed below. Find the five number summary for the data.

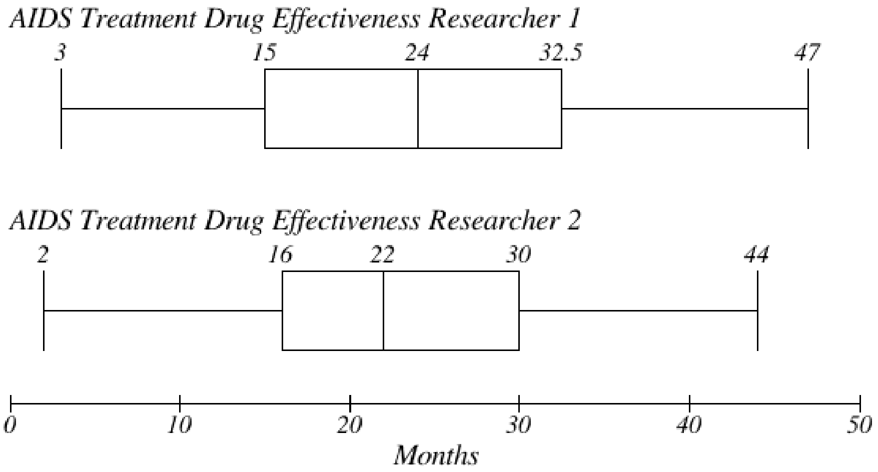
Income (thousands of dollars)	**Frequency**
15	2
20	11
25	16
30	9
35	4
40	2
45	4
50	1

Section 1.4 Exercises – Answer Key

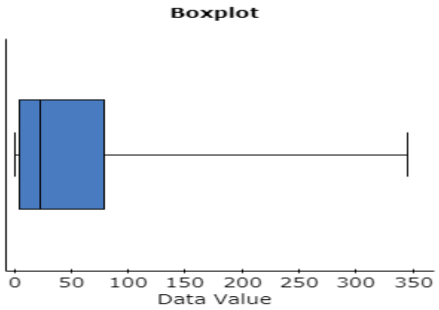
1. Kendra makes $90,000, and Kelsey makes $40,000, so Kendra makes $50,000 more.

2. 75%

3.



4.



5. 5.7

6. 10.9

7. a. answers vary. Example: {2,3,4,5,6}, {3,5,8,9,-4}, {2,6,7,-3,8}
   b. answers vary. Example: {2,3,4,5,6}, {3,5,8,9,-4}, {2,6,7,-3,8}
   c. answers vary. Example: {3,7,11,15,19}, {1,5,11,13,17}, {5,9,11,17,21}

8. Min: 15 Q1: 20 Q2: 25 Q3: 30 Max: 50

Section 1.5: Fitting Linear Models to Data

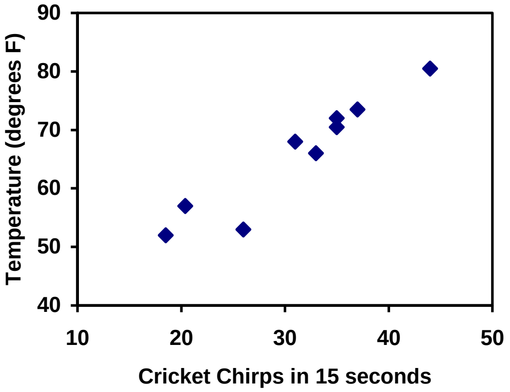
In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

Example 1

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit[3]. Plot this data, and determine whether the data appears to be linearly related.

chirps	44	35	20.4	33	31	35	18.5	37	26
Temp	80.5	70.5	57	66	68	72	52	73.5	53

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.



The simplest way to find an equation to approximate this data is to try to “eyeball” a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

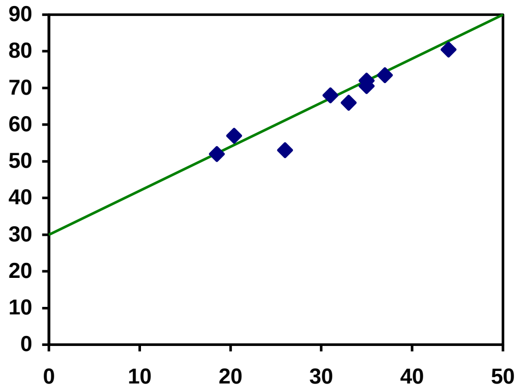
You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

---

[3] Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

Example 2

Using the table of values from the previous example, find a linear equation that fits the data by “eyeballing” a line that seems to fit.



On a graph, we could try sketching in a line. Note the scale on the axes have been adjusted to start at zero to include the vertical axis and vertical intercept in the graph.

Using the starting and ending points of our “hand drawn” line, points (0, 30) and (50, 90), this graph has a slope of $m = \frac{60}{50} = 1.2$ and a vertical intercept at 30, giving an equation of

$T = 30 + 1.2c$
where c is the number of chirps in 15 seconds, and T is the temperature in degrees Fahrenheit.

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.

Interpolation and Extrapolation

Interpolation: When we predict a value inside the domain and range of the data
Extrapolation: When we predict a value outside the domain and range of the data

For the temperature as a function of chirps in our hand drawn model above,

- Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.
- Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

Example 3

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model: $T = 30 + 1.2(30) = 66$ degrees.

Based on the data we have, this value seems reasonable.

The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c$$

$$10 = 1.2c$$

$$c \approx 8.33$$

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**.

Try it Now

1. What temperature would you predict if you counted 20 chirps in 15 seconds?

Fitting Lines with Technology

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[4]. This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators[5].

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[4] Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

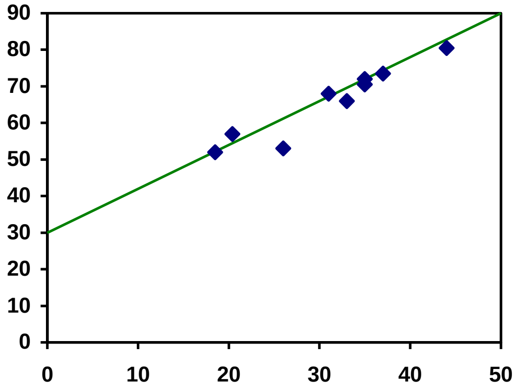
[5] For example, http://www.shodor.org/unchem/math/lls/leastsq.html

Example 4

Find the least-squares regression line using the cricket chirp data from above.

Using the cricket chirp data from earlier, with technology we obtain the equation:
$T = 30.281 + 1.143c$

Notice that this line is quite similar to the equation we "eyeballed", but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:
$T = 30.281 + 1.143(30) = 64.571 \approx 64.6$
degrees.



Most calculators and computer software will also provide you with the **correlation coefficient**, a measure of how closely the line fits the data.

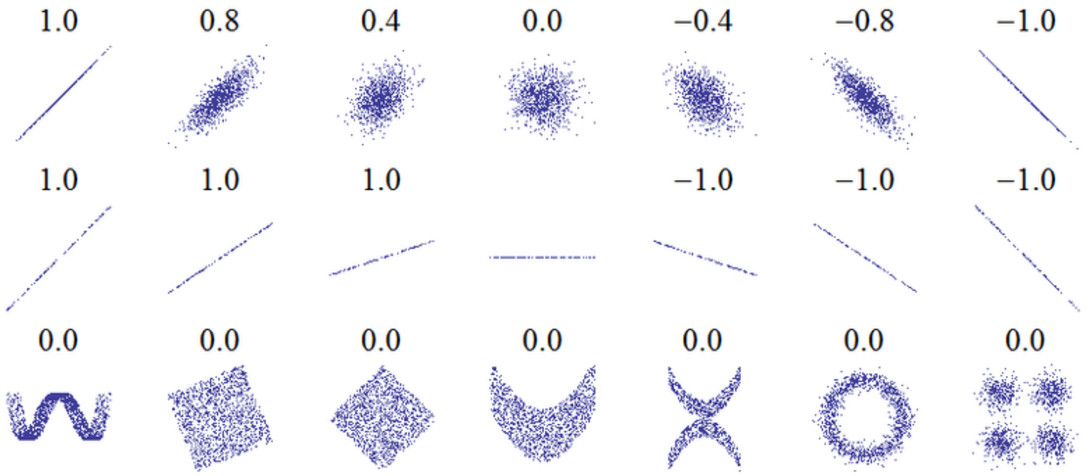
Correlation Coefficient

The correlation coefficient is a value, *r*, between -1 and 1.
***r* > 0 suggests a positive (increasing) relationship.**
***r* < 0 suggests a negative (decreasing) relationship.**
The closer the value is to 0, the more scattered the data are.
The closer the value is to 1 or -1, the less scattered the data are.

The correlation coefficient provides an easy way to get some idea of how close to a line the data falls.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of *r* and the graph of the data, here are some large data sets with their correlation coefficients:

Examples of Correlation Coefficient Values



[6]

Example 5

Calculate the correlation coefficient for our cricket data.

Because the data appears to follow a linear pattern, we can use technology to calculate $r = 0.9509$. Since this value is very close to 1, it suggests a strong increasing linear relationship.

Example 6

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.[7] Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

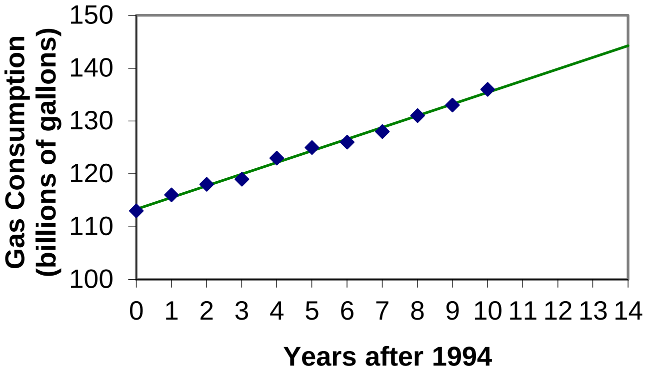
Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

---

[6] http://en.wikipedia.org/wiki/File:Correlation_examples.png

[7] http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

To make things simpler, a new input variable is introduced, *t*, representing years since 1994.



Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.
The least-squares regression equation is:
$C = 113.318 + 2.209t$.

Using this to predict consumption in 2008 (t = 14),
$C = 113.318 + 2.209(14) = 144.244$ billions of gallons

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

Try it Now

2. Use the model created by technology in example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Try it Now Answers

1. 54 degrees Fahrenheit
2. 150.871 billion gallons; extrapolation

Section 1.5 Exercises

1. The following is data for the first and second quiz scores for 8 students in a class. Plot the points, then sketch a line that fits the data.

First Quiz	11	20	24	25	33	42	46	49
Second Quiz	10	16	23	28	30	39	40	49

2. Eight students were asked to estimate their score on a 10 point quiz. Their estimated and actual scores are given. Plot the points, then sketch a line that fits the data.

Predicted	5	7	6	8	10	9	10	7
Actual	6	6	7	8	9	9	10	6

Based on each set of data given, calculate the regression line using your calculator or other technology tool, and determine the correlation coefficient.

3.

x	***y***
5	4
7	12
10	17
12	22
15	24

4.

x	***y***
8	23
15	41
26	53
31	72
56	103

5.

x	***y***
3	21.9
4	22.22
5	22.74
6	22.26
7	20.78
8	17.6
9	16.52
10	18.54
11	15.76
12	13.68
13	14.1
14	14.02
15	11.94
16	12.76
17	11.28
18	9.1

6.

x	***y***
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8
14	24.7
15	22
16	20.1
17	19.8
18	16.8

7. A regression was run to determine if there is a relationship between hours of TV watched per day (*x*) and number of sit-ups a person can do (*y*). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

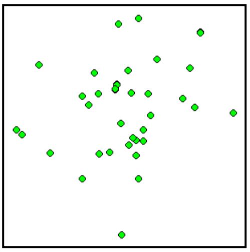
   y=ax+b
   a=-1.341
   b=32.234
   r^2=0.803
   r=-0.896

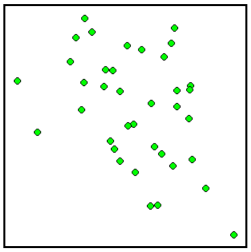
8. A regression was run to determine if there is a relationship between the diameter of a tree (*x*, in inches) and the tree's age (*y*, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

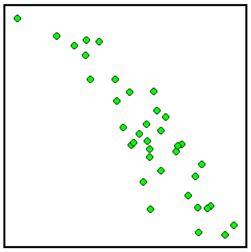
$y=ax+b$
$a=6.301$
$b=-1.044$
$r^2=0.940$
$r=-0.970$

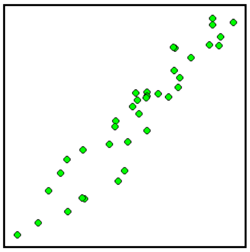
Match each scatterplot shown below with one of the four specified correlations.

9. $r = 0.95$ 10. $r = -0.89$ 11. $r = 0.26$ 12. $r = -0.39$

A


B


C


D


13. The US census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will the percentage exceed 35%?

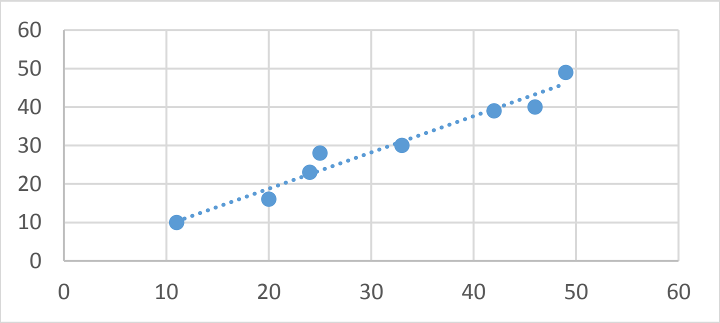
Year	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Percent Graduates	21.3	21.4	22.2	23.6	24.4	25.6	26.7	27.7	28	29.4

14. The US import of wine (in hectoliters) for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will imports exceed 12,000 hectoliters?

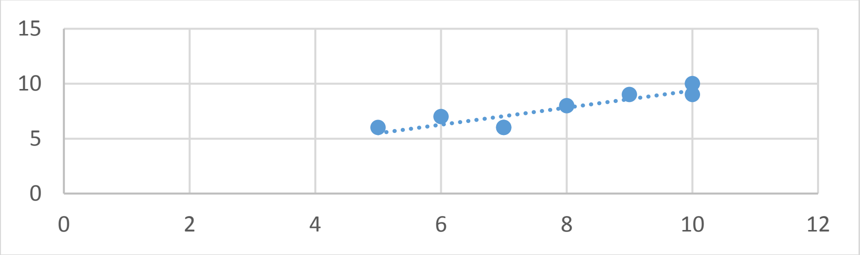
Year	1992	1994	1996	1998	2000	2002	2004	2006	2008	2009
Imports	2665	2688	3565	4129	4584	5655	6549	7950	8487	9462

Section 1.5 Exercises – Answer Key

1.



2.



3. $y = 1.971x - 3.519, \quad r = 0.967$

4. $y = 1.640x + 13.8, \quad r = 0.987$

5. $y = -0.902x + 26.04, \quad r = -0.968$

6. $y = -1.979x + 51.902, \quad r = -0.991$

7. $17.483 \approx 17$ situps

8. 61.966 years

9. D

10. C

11. A

12. B

13. Yes, the trend appears linear because r =0.994 and will exceed 35% near the end of the year 2019.

14. Yes, the trend appears linear because r =0.985 and will exceed 12,000 hectoliters near the end of the year 2016.

Chapter 2: Sets & Probability

Section 2.1: Sets

World View Note: Georg Ferdinand Cantor (1845-1918) gave birth to Set Theory. He compared magnitudes of infinite sets of numbers.

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets to understand relationships between groups and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in braces: { }.

Example 1

Some examples of sets defined by describing the contents:

a) The set of all even numbers
b) The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

a) {1, 3, 9, 12}
b) {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equal to the set {3, 1, 2}.

Notation

Commonly we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means "is an element of".

A set that contains no elements, { }, is called the **empty set** and is notated $\emptyset$.

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Brandon owns three Madonna albums. While Brandon's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$.

A **proper subset** is a subset that is not identical to the original set; it contains fewer elements.

If B is a proper subset of A, we write $B \subset A$.

Example 3

Consider these three sets: A = the set of all even numbers; $B = \{2, 4, 6\}$; and $C = \{2, 3, 4, 6\}$.

Here $B \subseteq A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since B is not identical to A.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A.

Example 4

Suppose a set contains the plays "Much Ado about Nothing", "MacBeth", and "A Midsummer's Night Dream". What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly, sets interact. For example, you and a new roommate decide to have a house party and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both).

The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.

The **complement** of a set A contains everything that is *not* in the set A. The complement is notated A', A^c, $\bar{A}$, or sometimes $\sim A$.

Example 5

Consider the sets: A = {red, green, blue}; B = {red, yellow, orange}; and C = {red, orange, yellow, green, blue, purple}.

a) Find $A \cup B$.

The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange}. Notice we only list red once.

b) Find $A \cap B$.

The intersection contains all the elements in both sets: $A \cap B$ = {red}.

c) Find $A^c \cap C$.

Here we are looking for all the elements that are *not* in set A and are also in C.
$A^c \cap C$ = {orange, yellow, purple}

Note that this could have been written as $A' \cap C$ or $\sim A \cap C$.

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context. A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A.

Example 6

a) If we were discussing searching for books, the universal set might be all the books in the library.
b) If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
c) If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers.

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic—to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}; F = {dog, cow, duck, pig, rabbit}; and W = {duck, rabbit, deer, frog, mouse}.

a) Find $(H \cap F) \cup W$.

We start with the intersection: $H \cap F$ = {dog, rabbit}.
Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}.

b) Find $H \cap (F \cup W)$.

We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}
Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}

c) Find $(H \cap F)^c \cap W$.

We start with the intersection: $H \cap F$ = {dog, rabbit}.
Now we want to find the elements of W that are *not* in $H \cap F$:
$(H \cap F)^c \cap W$ = {duck, deer, frog, mouse}.

Venn Diagrams

World View Note: To visualize the interaction of sets, in 1880 John Venn used overlapping circles, as he built on a similar idea used by Leonhard Euler in the 18th century. These illustrations are now called **Venn diagrams**.

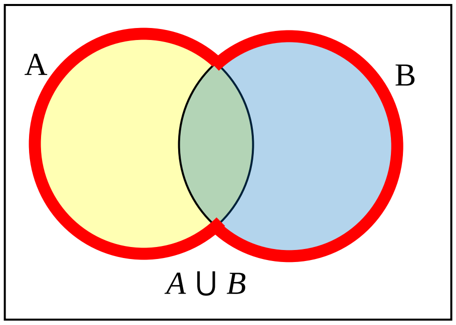
Venn Diagram
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

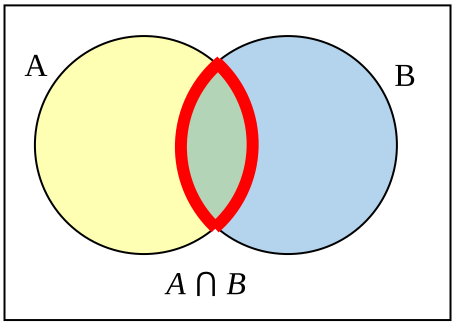
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

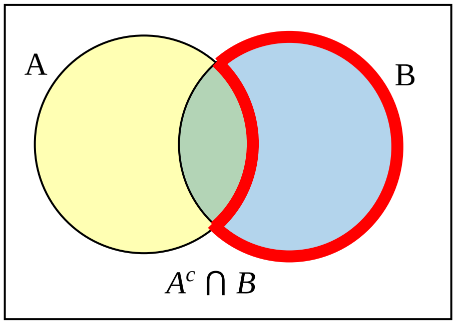
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets – in the overlap of the circles.



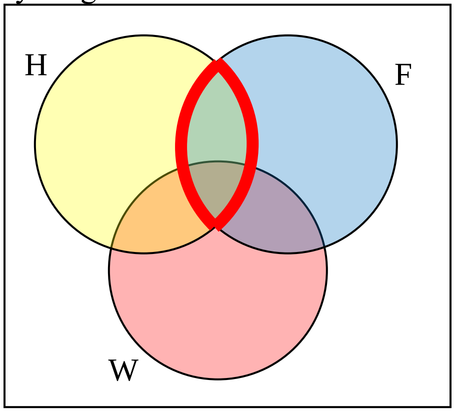
A^c will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



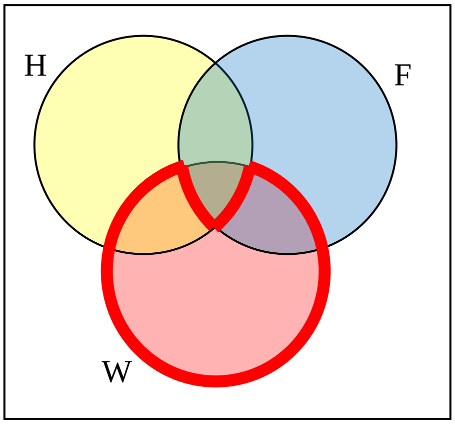
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$.

We'll start by identifying everything in the set $H \cap F$.

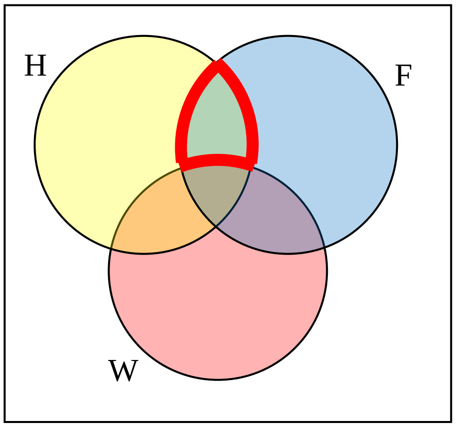


Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

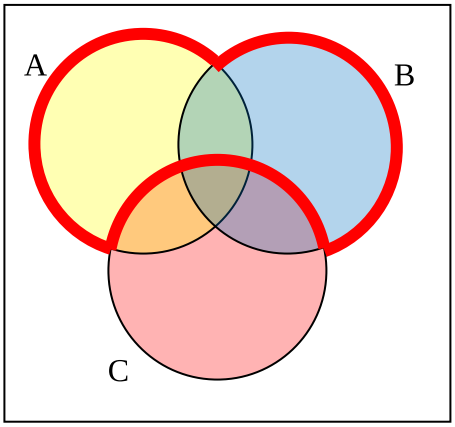
Create an expression to represent the outlined part of the Venn diagram shown.



The elements in the outlined set *are* in sets *H* and *F*, but are not in set *W*. So we could represent this set as $H \cap F \cap W^c$.

Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown.



Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality
The number of elements in a set is the cardinality of that set. The cardinality of the set *A* is often notated as $|A|$ or $n(A)$.

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$. What is the cardinality of B? $A \cup B$, $A \cap B$?

The cardinality of B is 4, since there are 4 elements in the set. We can write $n(B)=4$.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements. We can write $n(A \cup B)=7$.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements. We can write $n(A \cap B)=3$.

Example 13

If P is the set of English names for the months of the year, determine $n(P)$.

The cardinality of this set is 12, since there are 12 months in the year. Therefore, $n(P)=12$.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

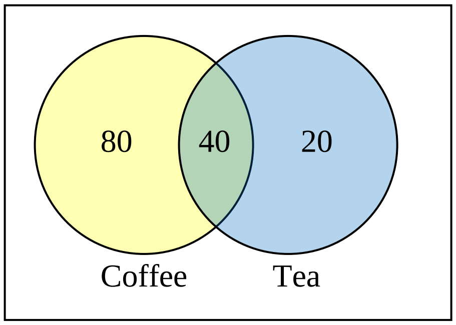
Example 14

A survey asks 200 people "What beverage do you drink in the morning?", and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea nor coffee?

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.



We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.
200 – 20 – 80 – 40 = 60 people who drink neither.

Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter nor Facebook?

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols:

$$\mathrm{n}(F \cup T) = \mathrm{n}(F) + \mathrm{n}(T) - \mathrm{n}(F \cap T)$$
$$\mathrm{n}(F \cup T) = 70\% + 40\% - 20\% \ = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T$ = 90%, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties.

Cardinality properties

$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B)$ and

$\mathrm{n}(A^c) = \mathrm{n}(U) - \mathrm{n}(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$\mathrm{n}(A \cap B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cup B).$$

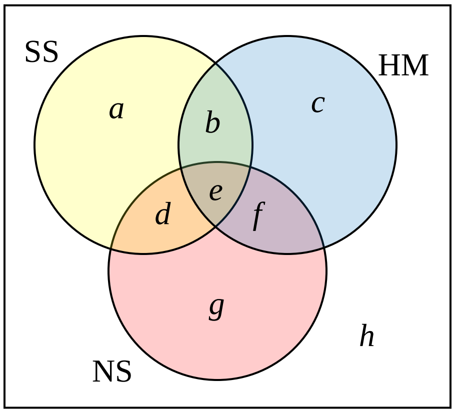
Example 16

Fifty students were surveyed and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course
19 were taking a NS course
7 were taking SS and NS
3 were taking all three
26 were taking a HM course
9 were taking SS and HM
10 were taking HM and NS
7 were taking none

How many students are only taking a SS course?

It might help to look at a Venn diagram.
From the given data, we know that there are
3 students in region *e* and
7 students in region *h*.



Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be
7 – 3 = 4 students in region *d*.

Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be
10 – 3 = 7 students in region *f*.

Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b*.

Now, we know that 21 students were taking a SS course. This includes students from regions *a*, *b*, *d*, and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a*.

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in none of these things?

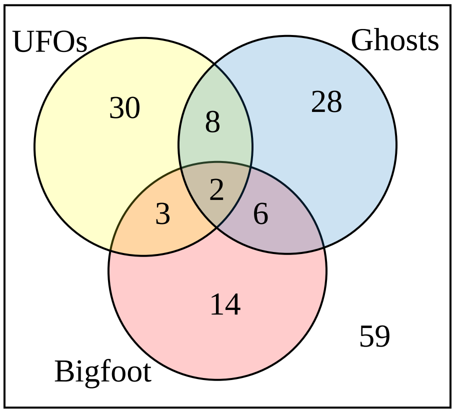
Try it Now Answers

1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2. $A \cup C$ = {red, orange, yellow, green, blue purple}
$B^c \cap A$ = {green, blue}

3. $A \cup B \cap C^c$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves 150 – 91 = 59 who believe in none.



Section 2.1 Exercises

1. Using set notation, list the elements of the set “The letters of the word Mississippi”.

2. Using set notation, list the elements of the set “Months of the year”.

3. Write a verbal description of the set: {3, 6, 9}.

4. Write a verbal description of the set: {a, i, e, o, u}.

5. Is {1, 3, 5} a subset of the set of odd integers?

6. Is {A, B, C} a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.
A = {1, 2, 3, 4, 5} B = {1, 3, 5} C = {4, 6} U = {0,1,2,3,…,10}

7. $3 \in B$ 8. $5 \in C$ 9. $B \subset A$ 10. $C \subset A$ 11. $C \subset B$ 12. $C \subset A$

Using the sets from above, and treating U as the universal set, find each of the following:

13. $A \cup B$ 14. $A \cup C$ 15. $A \cap C$ 16. $B \cap C$ 17. A^c 18. B^c

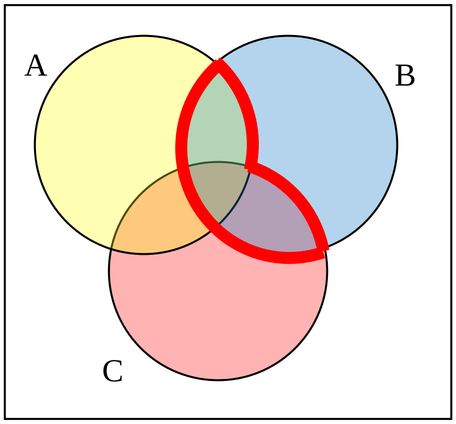
Consider the sets D = {b, a, c, k}, E = {t, a, s, k}, and F = {b, a, t, h}. Using these sets, find the following:

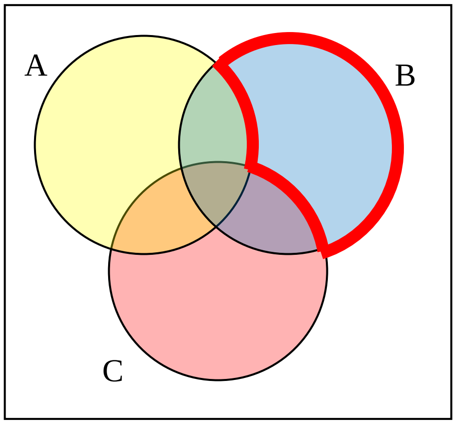
19. $D^c \cap E$ 20. $F^c \cap D$ 21. $(D \cap E) \cup F$
22. $D \cap (E \cup F)$ 23. $(F \cap E)^c \cap D$ 24. $(D \cup E)^c \cap F$

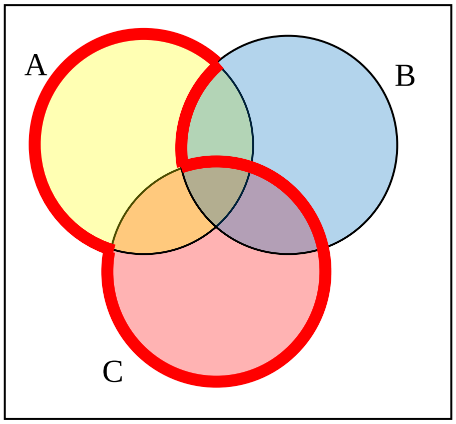
Create a Venn diagram to illustrate each of the following:

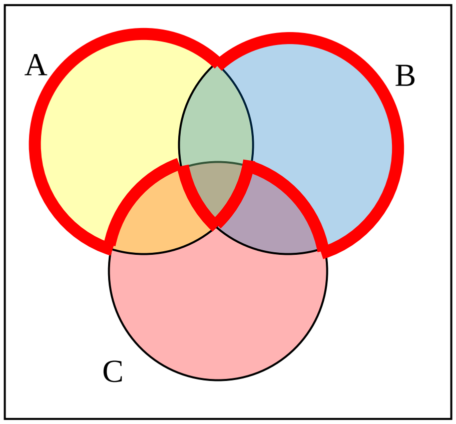
25. $(F \cap E) \cup D$ 26. $(D \cup E)^c \cap F$
27. $(F \cap E) \cap D$ 28. $(D \cup E) \cup F$

Write an expression for the shaded region.

29. 

30. 

31. 

32. 

Consider the sets $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$, and $C = \{4, 6\}$. Find the cardinality of the given set.

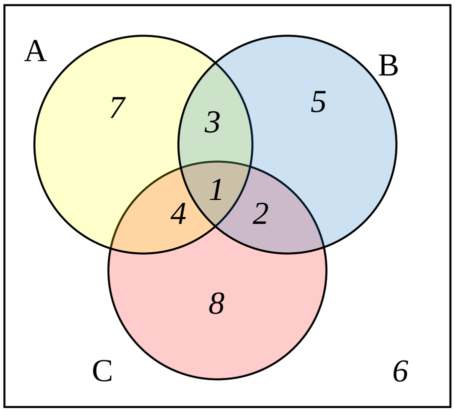
33. $n(A)$

34. $n(B)$

35. $n(A \cup C)$

36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $n(A \cap C)$

38. $n(B \cup C)$

39. $n(A \cap B \cap C^c)$

40. $n(A \cap B^c \cap C)$

41. If $n(G) = 20$, $n(H) = 30$, $n(G \cap H) = 5$, find $n(G \cup H)$

42. If $n(G) = 5$, $n(H) = 8$, $n(G \cap H) = 4$, find $n(G \cup H)$

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 use only Netflix
24 use only a video store
48 use only Netflix and Redbox
10 use all three
62 use only Redbox
16 use only a video store and Redbox
30 use only a video store and Netflix
25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color
16 only said brand
42 said only color and brand
102 said all three
8 only said size
20 said only color and size
53 said only size and brand
20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

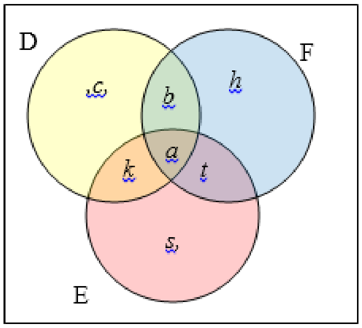
18 had seen *The Matrix* (*M*)
20 had seen *Lord of the Rings (LotR)*
14 had seen *LotR* and *SW*
6 had seen all three
24 had seen *Star Wars (SW)*
10 had seen *M* and *SW*
12 had seen *M* and *LotR*

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don’t use any alternate transportation.

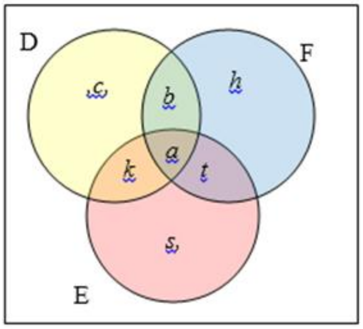
30% use the bus
25% walk
10% ride a bicycle and walk
2% use all three
20% ride a bicycle
5% use the bus and ride a bicycle
12% use the bus and walk

Section 2.1 Exercises – Answer Key

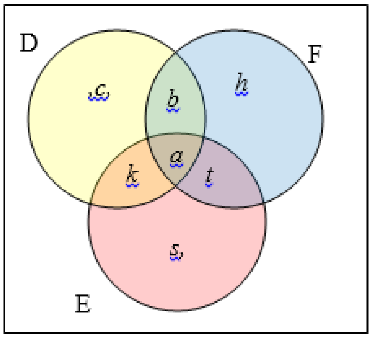
1) {M,i,s,p}
2) {January, February, March, April, May, June, July, August, September, October, November, December}
3) The set of all multiples of 3 from 3 through 9.
4) Set of all letters that are always a vowel.
5) yes
6} yes
7) true
8) false
9) false
10) false
11) false
12) true
13) {1,2,3,4,5}
14) {1,2,3,4,5,6}
15) {4}
16) {}
17) {6,7,8,9,10}
18) {2,4,6,7,8,9,10}
19) {t,s}
20) {b,c,k}
21) {b,a,t,h,k}
22) {a,b,k}
23) {b,k,c}
24) {h}
25) {t,a,b,k,c}



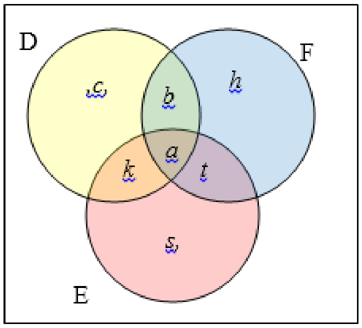
26) {h}



27) {a}



28) {a,b,c,d,e,f,g,h}



29) $(A \cap B)\ U\ (B \cap C)$
30) $(A\ U\ C)^{c} \cap B$
31) $(B^{c} \cap A)\ U\ C$
32) $(C^{c} \cap (A\ U\ B))\ U\ (A \cap B \cap C)$
33) 5
34) 3
35) 6
36) 1
37) 5
38) 23
39) 3
40) 4
41) 45
42) 9
43) 136
44) 213
45) a) 8 b) 6
46) a) 15 b) 50

Section 2.2: Basis Concepts of Probability

Introduction

World View Note: In the 17th century, Blaise Pascal and Pierre de Fermat worked on calculations of probabilities and they supplied vital links of the reasoning chain for probability theory. Two roots of origin of probability theory are solutions of wagering problems and processing of statistical data using mortality tables.

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time. We would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," he or she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**. An **event** is any particular outcome or group of outcomes. A **simple event** is an event that cannot be broken down further. The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: {1, 2, 3, 4, 5, 6}.

Some examples of simple events:
We roll a 1.
We roll a 5.

Some compound events:
We roll a number bigger than 4.
We roll an even number.

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event *E* using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally likely outcomes}}.$$

Example 2

If we roll a 6-sided die, calculate
a) P(rolling a 1)
b) P(rolling a number bigger than 4)

Recall that the sample space is {1,2,3,4,5,6}

a) There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$

b) There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet, and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones.

(The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now 1

At some random moment, you look at your clock and note the minutes reading.
a. What is probability the minutes reading is 15?
b. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

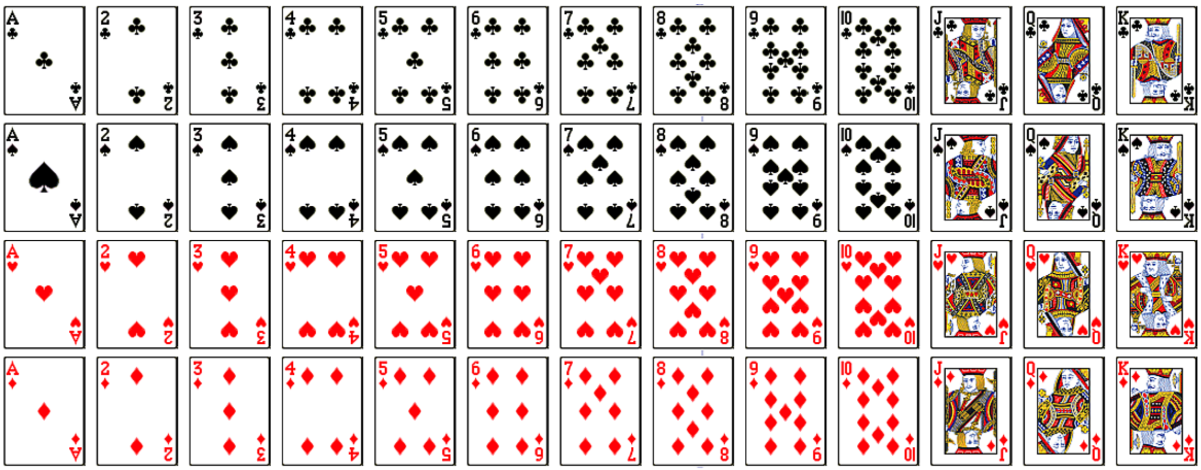


Image obtained from http://www.milefoot.com/math/discrete/counting/images/cards.png.

World View Note: Gaming and gambling with a dice and cards are sources that gave birth and development to probability. These sources were for amusement and profit and to boost soldiers' morale. In addition, they were favored by the Greeks and the Romans but forbidden by the Jews.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

There are 52 cards in the deck and 4 Aces so $P(Ace)=\frac{4}{52}=\frac{1}{13}\approx 0.0769$

We can also think of probabilities as percentages: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0, if there are no outcomes that correspond with the event. The largest possible probability is 1, if all possible outcomes correspond with the event.

Certain and Impossible events

An impossible event has a probability of 0.

A certain event has a probability of 1.

The probability of any event must be $0 \le P(E) \le 1$

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work*.

Section 2.2 Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
    a. A red ball is drawn
    b. A white ball is drawn
2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
    a. A consonant
    b. A vowel
3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.
4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Compute the probability of tossing a six-sided die and getting a 7.
7. After giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.
10. Compute the probability of tossing a six-sided die and getting a number less than 3.
11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?
12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?
13. Compute the probability of rolling a 12-sided die and getting a number other than 8.

Section 2.2 Exercises – Answer Key

1) a) 6/13 b) 2/13
2) a) 11/14 b) 3/13
3) 30/67
4) 41/81
5) 1/6
6) 0/6 = 0
7) 26/65
8) 1/3
9) 1/2
10) 1/3
11) 1/13
12) 1/4
13) 11/12

Section 2.3: Working with Events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six})=\frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six})+P(\text{not a six})=\frac{1}{6}+\frac{5}{6}=\frac{6}{6}=1.$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining n - m outcomes correspond to E not happening, thus $P(\text{not } E)=\frac{n-m}{n}=\frac{n}{n}-\frac{m}{n}=1-\frac{m}{n}=1-P(E)$.

Complement of an Event

The **complement** of an event is the event "*E* doesn't happen"

The notation $\bar{E}$ is used for the complement of event *E*.

We can compute the probability of the complement using $P\left(\bar{E}\right)=1-P(E)$.

Notice also that $P(E)=1-P\left(\bar{E}\right)$.

Example 1

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

There are 13 hearts in the deck, so $P(\text{heart})=\frac{13}{52}=\frac{1}{4}$.

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart})=1-P(\text{heart})=1-\frac{1}{4}=\frac{3}{4}.$$

Probability of Two Independent Events

Example 2

Suppose we flipped a coin and rolled a die, and want to know the probability of getting a head on the coin and a 6 on the die.

We could list all possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}. Notice there are 2 · 6 = 12 total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 3

Are these events independent?

a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

c) You draw a card from a deck, then draw a second card without replacing the first.

Answers:

a) The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

P(A and B) for independent events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring.

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 4

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$.

Try it Now 2

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 5

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Here, there are still 12 possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both—we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

***P*(*A* or *B*)**

The probability of either *A* or *B* occurring (or both) is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Example 6

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 7

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so $P(\text{red}) = \frac{26}{52}$.

There are four kings, so $P(\text{King}) = \frac{4}{52}$.

There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$.

We can then calculate:

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}.$$

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example 8

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a red car *and* got a speeding ticket
b) Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:
P(had a red car) + P(got a speeding ticket) – P(had a red car and got a speeding ticket)

$$= \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}.$$

Section 2.3 Exercises

1. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?

2. After giving a test to a group of students, the grades and gender are summarized below. What is the probability that a student chosen at random did not earn a C?

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

3. The following table shows the number of credit cards owned by a group of individuals. What is the probability that a person chosen at random has at least one credit card?

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

4. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?

5. A fair coin is flipped twice. What is the probability of showing heads on both flips?

6. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?

7. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?

8. If you pull a random card from a deck of playing cards, what is the probability it is not a diamond?

9. Are these events independent?
   a) A fair quarter is tossed two times. The two events are (1) first toss is a tail and (2) second toss is a tail.
   b) You draw a card from a deck, then draw a second card without replacing the first.

10. Suppose we draw one card from a standard deck. What is the probability that we get a Jack or an Ace?

11. Suppose we draw one card from a standard deck. What is the probability that we get a black card or a Queen?

12. Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a tail on the coin or a 4 on the die.

Section 2.3 Exercises – Answer Key

1) 51/52
2) 8/13
3) 2/3
4) 1/36
5) 1/4
6) 1/12
7) 0.0441 (4.41%)
8) 3/4
9) a) yes, independent b) no, dependent
10) 2/13
11) 7/13
12) 1/12

Section 2.4: Conditional Probability

World View Note: Christian Huygens (1629 – 1695) wrote the first text on probability theory. In addition, he worked with his brother, Ludwig, on John Graunt's figures and got the mortality curve produced from statistical data.

Often it is required to compute the probability of an event given that another event has occurred.

Example 1

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52}\cdot\frac{4}{52}=\frac{1}{169}$. However, this would be incorrect, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:
P(ace on second draw | an ace on the first draw).

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51}=\frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52}\cdot\frac{3}{51}=\frac{12}{2652}=\frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as $P(B \mid A)$. This is read as "the probability of B given A".

Example 2

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 3

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket, *given* they have a red car.

b) Has a red car, *given* they have a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

P(ticket | red car) = $\frac{15}{150} = \frac{1}{10} = 0.1$.

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

P(red car | ticket) = $\frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that P(B | A) is **not** equal to P(A | B).

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then

$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

Example 4

If you pull 2 cards out of a deck, what is the probability that both are spades?

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck and one less card total.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Example 5

If you draw two cards from a deck, what is the probability that you will get the ace of diamonds and a black card?

You can satisfy this condition by having Case A or Case B, as follows:
Case A) you can get the ace of diamonds first and then a black card or
Case B) you can get a black card first and then the ace of diamonds.

Let's calculate the probability of Case A. The probability that the first card is the ace of diamonds is $\frac{1}{52}$. The probability that the second card is black, given that the first card is the ace of diamonds, is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the ace of diamonds, given that the first card is black, is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is *P*(A) + *P*(B) - *P*(A and B). In this problem, *P*(A and B) = 0 since the first card cannot be the ace of diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102}$. The probability that you will get the ace of diamonds and a black card when drawing two cards from a deck is $\frac{2}{102} = \frac{1}{51}$.

Try it Now 1

In your drawer you have 10 socks, 6 of which are white. If you reach in and randomly grab two socks, what is the probability that both are white?

Example 6

A home pregnancy test was given to women and then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find:
a) *P*(not pregnant | positive test result)
b) *P*(positive test result | not pregnant)

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

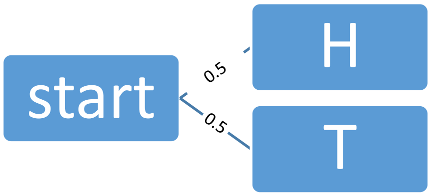
a) Since we know the test result was positive, we are limited to the 75 women in the first column, of which 5 were not pregnant. $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

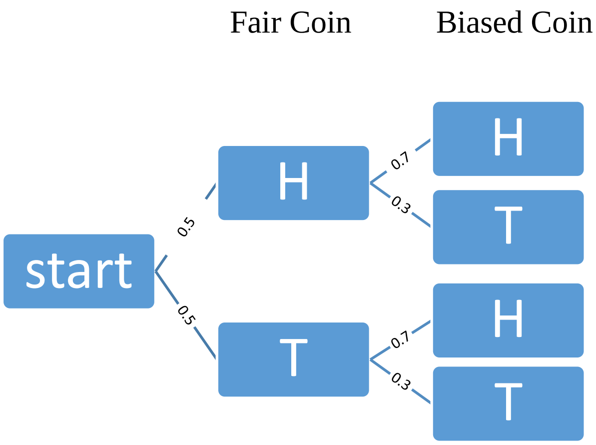
Tree Diagrams

Tree diagrams are used to display the different outcomes for an experiment and the probabilities of those outcomes. Each outcome is represented by a branch extending from a point. For example, the tree diagram for flipping a fair coin is below. Since there are two outcomes, heads H and tails T, the tree has two branches.



The branches are drawn extending to the right. This is our choice. The branches may extend in any direction. The end of each branch is marked with a small black dot called a **node**. Each node is labeled with an outcome. The center of each branch is labeled with the probability of that outcome.

Suppose a second experiment is performed after this. We present its outcomes by drawing additional branches starting at the rightmost nodes above and extending to the right. For example, suppose we flip a biased coin after we flip the fair one. The probability of heads with the biased coin is 0.7 (so the probability of tails is 1 – 0.7 = 0.3). The tree diagram for flipping the two coins follows.



Outcome	Probability
HH	(0.5)(0.7) = 0.35
HT	(0.5)(0.3) = 0.15
TH	(0.5)(0.7) = 0.35
TT	(0.5)(0.3) = 0.15
	Total = 1.00

Begin at the start point on the far left. The first set of two branches corresponds to outcomes for the fair coin. Each path we can trace from the start point to an end node on the far right gives a different outcome. For example, HT means we get H on the fair coin and T on the biased coin. Verify that there are four different paths; therefore, we have four outcomes in all.

To get the probability of an outcome, multiply the probabilities on each branch along the path ending in that outcome. For example, the probability of the outcome HH is (0.5)(0.7) = 0.35. The four outcomes listed above are ***mutually exclusive*** (if one occurs then other three cannot occur), and ***exhaustive*** (there are no other outcomes). To verify this, add the four probabilities for the four outcomes. We get 1.00 or certainty. When we flip the two coins, one of the four outcomes is certain to occur.

The probability of different sets of outcomes is found by adding their corresponding probabilities listed on the right side. For example, the probability of getting two heads HH or two tails TT is found by adding those two probabilities.

P(HH **OR** TT) = 0.35 + 0.15 = 0.5

The probability of getting at least one head is given by three outcomes HH, HT, and TH. These are mutually exclusive events, so the probability is found by adding the three corresponding probabilities.

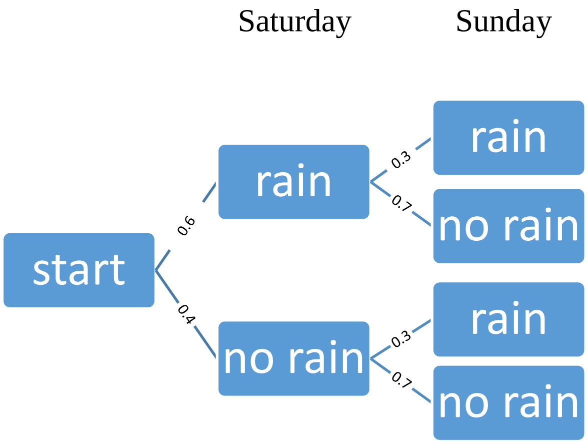
P(HH **OR** HT **OR** TH) = 0.35 + 0.15 + 0.35 = 0.85

Probability of Outcomes Using a Tree Diagram
To get the probability of one outcome, multiply the probabilities along the path ending in that outcome.
To get the probability of a set of outcomes, add the corresponding probabilities of each outcome in the set.

Example 7

A meteorologist says the chance of getting rain on Saturday is 60% and the chance of getting rain on Sunday is 30%. Based on this report, what is the probability we get some rain over the weekend.

First, build a tree diagram for the weekend forecast. The first set of branches corresponds to Saturday. The second set corresponds to Sunday.



Outcome	Probability
Rain, rain	(0.6)(0.3) = 0.18
Rain, no rain	(0.6)(0.7) = 0.42
No rain, rain	(0.4)(0.3) = 0.12
No rain, no rain	(0.4)(0.7) = 0.28
	Total = 1.00

Since the probability of rain on Saturday is 60%, the probability of no rain on Saturday is 40%. Since the probability of rain on Sunday is 30%, the probability of no rain on Sunday is 70 %. There are four outcomes. Each outcome is the pair Saturday, Sunday. For example, the outcome “rain, no rain” means we get rain on Saturday, but no rain on Sunday. Verify that the sum of the four probabilities is 1.

Second, find the probability of getting some rain on the weekend. Since 3 out of the 4 outcomes results in some rain on the weekend, we add the three corresponding probabilities for our answer.

P(some rain on the weekend) = P(rain, rain **OR** rain, no rain **OR** no rain, rain)
= 0.18 + 0.42 + 0.12
= 0.72

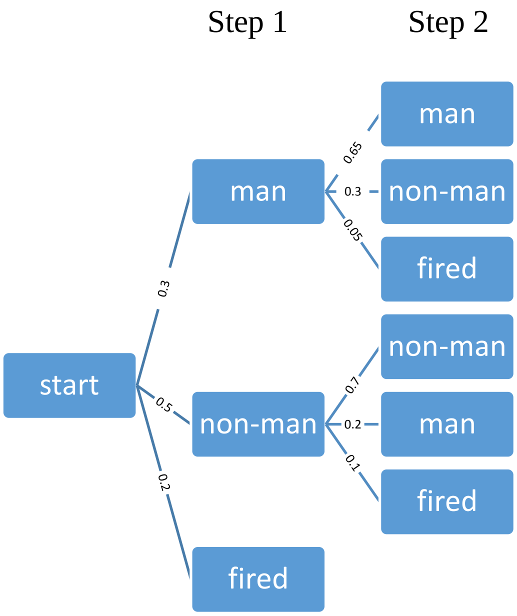
There is a 72% chance of getting some rain on the weekend. We can check our answer by using the ***complement of an event***. The sum of the probabilities is 1. The probability of no rain on either day (no rain, no rain) is 0.28. So the probability of some rain is P(some rain) = 1 – P(no rain, no rain) = 1 – 0.28 = 0.72.

A tree diagram has the structure of a flow chart.

Example 8

A corporate training program is used to direct hires to jobs suiting their skills. It is a two-step program. Step 1 identifies 30% of the hires as management trainees, 50% as non-management trainees, and 20% to be fired. In Step 2, 65% of the management trainees are assigned to management positions, 30% to non-management positions, and 5% are fired. Also in Step 2, 70% of the non-management trainees are kept in that category, 20% are assigned to management positions, and 10% are fired.

a) Construct a tree diagram for this remaining program.



Outcome	Probability
man, man	(0.3)(0.65) = 0.195
man, non-man	(0.3)(0.3) = 0.09
man, fired	(0.3)(0.05) = 0.015
non-man, non-man	(0.5)(0.7) = 0.35
non-man, man	(0.5)(0.2) = 0.10
non-man, fired	(0.5)(0.1) = 0.05
fired	= 0.20
	Total = 1.00

Each rightmost node defines a different outcome. There are 7 possible outcomes for the training program. Verify that the sum of the 7 probabilities is 1.

b) What is the probability that a randomly chosen hire will be assigned to a management position at the end of the training period?

Add the probability for each outcome ending in a management position.

P(assigned to management) = P(man, man, **OR** non-man, man)
= 0.195 + 0.10
= 0.295

There is a 29.5% chance a randomly selected hire will be assigned to a management position at the end of the training period.

c) What is the probability that a randomly chosen hire will be fired by the end of the training period?

Add the probabilities for the outcomes ending in being fired. There are three of them.

P(fired) = P(man, fired **OR** non-man, fired **OR** fired)
= 0.015 + 0.05 +0.20
= 0.265

There is a 26.5% chance a randomly selected hire will be fired.

d) What is the probability that a randomly chosen hire will start as a non-management trainee, but will not be assigned to a non-management position?

The outcomes non-man, man and non-man, fired correspond to starting as a non-management trainee, but not being assigned to a non-management position.

P(non-man, man **OR** non-man, fired) = 0.10 + 0.05
= 0.15

The probability is 15%.

Try it Now Answer

1. $\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$

Section 2.4 Exercises

1. A jar contains five red marbles numbered 1 to 5 and eight blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Even-numbered given that the marble is red.
    b. Red given that the marble is even-numbered.

2. A jar contains four red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Odd-numbered given that the marble is blue.
    b. Blue given that the marble is odd-numbered.

3. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

4. Find the probability of rolling a 1 on a fair die, given that the last three rolls were all ones.

5. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

6. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

7. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive."
    a. Find the probability that a person has the virus given that they have tested positive; that is, find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative; that is, find P(not A | not B).

8. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive; that is, find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative; that is, find P(not A | not B).

9. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.

10. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

11. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?

12. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

For ***exercises 13-35***, build a tree diagram for each problem. Then, use it to answer each problem.

13. A meteorologist says the chance of getting rain on Saturday is 50% and the chance of getting rain on Sunday is 50%. Based on her report, what is the probability we get some rain over the weekend?

14. A meteorologist says the chance of getting rain on Saturday is 80% and the chance of getting rain on Sunday is 60%. Based on his report,
    a. What is the probability we get some rain over the weekend?
    b. What is the probability we get no rain?

15. A meteorologist says the chance of getting rain on Monday is 40%, on Tuesday is 20% and on Wednesday is 30%. Based on his report,
    a. What is the probability we get some rain over the three-day period?
    b. What is the probability we get no rain?

16. Samantha the Seer says the chance of getting rain on Wednesday is 50%, on Thursday is 50%, and on Friday is 50%. If Samantha’s perception is correct, what is the probability we get some rain over the three-day period?

17. An automobile repair shop uses a two-step diagnostic procedure to repair carburetors. Step 1 locates the problem in the carburetor with probability 0.8. Step 2, which is used only if Step 1 fails to locate the problem, locates the problem with probability 0.6.
    a. What is the probability that the procedure will locate the problem?
    b. What is the probability that the procedure will fail to locate the problem?

18. An urn contains five balls, each labeled with a different number 1 through 5. We randomly select a ball from the urn without replacement. If the number on the ball is 4, we stop. If it is not 4, we randomly select a second ball from the urn. We look at it to see if it is a 4 or not. Then, the game stops. What is the probability that one of the two selected balls is a 4?

19. An urn contains ten balls, five Blue, three Red, and two White. We randomly select a ball from the urn without replacement. If the ball is White, we stop. If the ball is not White, we randomly select a second ball from the urn. If the ball is White, we stop. If the ball is not White, we randomly select a third ball from the urn.
    a. What is the probability none of the three selected balls is White?
    b. What is the probability the second ball selected is White?
    c. What is the probability at least one of the three balls is White?

20. We toss a single fair die. If a 1 comes up, we win. If not, we toss the die a second time. If a 1 comes up on the second toss, we win. Otherwise, we lose. What is the probability we win?

21. There are three defective bulbs in a box of 12. We randomly select one bulb without replacement, and test it. If it is defective, we throw away the box of bulbs. If not, we select a second bulb. If it is defective, we throw away the box of bulbs. If not, we ship the box to the consumer. What is the probability we throw away the box of bulbs?

22. A well shuffled deck of cards is placed face down on a table next to a pile of money. You randomly select a card without replacement. If it is a face card, you win the money. If not, you select a second card. If the second card is a face card, you win the money. Otherwise, you lose. What is the probability you win the money?

23. A corporate training program is used to direct hires to jobs suiting their skills. It is a two-step program. Step 1 identifies 20% as management trainees, 70% as non-management trainees, and 10% to be fired. In Step 2, 60% of the management trainees are assigned to management positions, 30% to non-management positions, and 10% are fired. In Step 2, 75% of the non-management trainees are kept in that category, 20% are assigned to management positions, and 5% are fired.
    a. What is the probability that a randomly chosen hire will be assigned to a management position at the end of the training period?
    b. What is the probability that a randomly chosen hire will be fired by the end of the training period?
    c. What is the probability that a randomly chosen hire will start as a non-management trainee, but will not be assigned to a non-management position?

24. A blue urn contains two blue balls and one red ball. A red urn contains one blue ball and three red balls. We randomly select a ball from the blue urn and note its color. Then, without replacing the first ball, we randomly select a second ball from the urn having the same color as the first ball. What is the probability that the second ball is red?

25. On the standard roulette wheel in the U. S., 18 numbers are red, 18 are black, and 2 are green. You wager $1 on red (that is, you bet that one of the 18 red numbers will come up). If red comes up, you win $1 (you also get the $1 back that you wagered). If not, you lose your $1 wager. If you lose, you then double your last bet, and put $2 on red. If red comes up, you win $2 (you actually recover the $1 lost in the first bet, which means you are now up $1; you also get the $2 back that you wagered). If not, you lose your $2 wager. If you lose, you then double the previous bet, and put $4 on red. If red comes up,

you win \$4 (you recover the \$1 and \$2 wagers lost in the first two spins of the wheel, and you are now up \$1; you also get back your \$4 wager). If not, you lose your \$4 wager. If you win on any of the three spins, the game ends. If you go three spins without winning, the game ends.

a. What is the probability you win in three spins of the wheel?
b. What is the probability you don't win in the three spins?

26. Repeat problem 25, but this time with five spins of the roulette wheel. (You wager \$8 on the fourth spin, and \$18 on the fifth.)
Discussion: Will doubling up always result in your being a winner?

27. A card is randomly drawn from a 52-card deck. If the card is a face card, we toss a coin. If the card is not a face card, we toss a die.
    a. What is the probability we end the game with a 4 on the die?
    b. What is the probability we end the game with Tails on the coin?

28. A card is drawn from a 52-card deck without replacement. We continue to draw cards until we draw an ace, or until we have drawn four cards, whichever comes first. What is the probability that the drawing ends before the fourth draw?

29. You are in a game show, and you have a choice of three doors. Behind one door is an all paid vacation for two to Tahiti. Behind the other two doors are booby prizes. Your host, Monte Hall, knows what is behind each door. You pick one of the three doors, but you are not shown what is behind it. Instead, Monte opens one of the other two doors to intentionally show you one of the booby prizes. Monte then offers you a choice: to stay with the first door that you choose or to switch your choice to the other closed door. Should you stick with the original door you chose or switch to the other door? Which gives you the greatest chance of winning the grand prize, or doesn't it matter? (Hint: Build a tree diagram to model the problem,)

30. Craps is a popular casino game played with a pair of dice. The shooter (the customer) is given the dice and plays against the house. The shooter throws the two dice. If a sum of 7 or 11 comes up (called a ***natural***), the shooter wins, and the game is over. If a sum of 2, 3, or 12 comes up (called ***craps***), the shooter loses, and the game is over. If a sum of 4, 5, 6, 8, 9, or 10 comes up, this is called your ***point***. The shooter picks up the dice again, and keeps rolling them until he either gets his point (he wins), or he gets a 7 (he loses), whichever comes first. If a sum other that the point or a 7 comes up, that roll is ignored and the shooter picks up the dice again and keeps rolling until the point or 7 comes up, which will then end the game. What is the probability the shooter wins the game? (Hint: Build a tree diagram. Begin with branches from each sum 2 through 12 on the first roll. You will then extend six additional sets of branches for the six possible points. The probabilities you label on these point branches are not based on 36 possible outcomes, but on subsets of these outcomes!)

31. Assuming that boy and girl babies are equally likely, build a gender tree diagram for a family planning to have four children.
    a. What is the probability the oldest child is a girl?
    b. What is the probability of having two boys and two girls?

   c. What is the probability of having a child of each gender in the group of four?

32. A drawer contains four red socks and six black socks. You wake up blurry-eyed one morning, open the drawer, and randomly select two socks. What is the probability both are the same color?

33. A bag contains three blocks numbered 1, 2, and 3. You reach into the bag and randomly select a block. You look at the number on the block and flip a fair coin that number of times. What is the probability of getting heads on at least one toss?

34. A little girl has two small bags of jelly beans. Each bag contains two red jelly beans and three green ones. She likes the green ones best. If she randomly selects one jelly bean from each bag, what is the chance she gets at least one green one?

35. The following comes from the *Ask Marilyn* column. A write says: “I recently returned from a trip to China, where the government is so concerned about population growth that is has instituted strict laws about family size. In the cities, a couple is permitted to have only one child. In the countryside, where sons traditionally have been valued, if the first child is a son, the couple may have no more children. But if the first child is a daughter, the couple may have another child. Regardless of the sex of the second child, no more are permitted. How will this policy affect the mix of males and females?” Build a tree diagram for the countryside births. Then, apply the probabilities labeled on the branches to a group of 100 newborn children.

Section 2.4 Exercises – Answer Key

1) a) $\frac{2}{5}=0.4$ b) $\frac{2}{6}\approx 0.33$
2) a) $\frac{1}{2}=0.5$ b) $\frac{2}{3}\approx 0.667$
3) $\frac{1}{2}=0.5$
4) $\frac{1}{6}\approx 0.0167$
5) $\frac{3}{14}\approx 0.2143$
6) $\frac{2}{5}=0.4$
7) a) 0.0221 b) 0.9997
8) a) 0.02206 b) 0.99972
9) 0.06604
10) 0.0298
11) 0.09395
12) 0.0001
13) 0.75 or 75%
14) a) 0.92 or 92% b) 0.08 or 8%
15) a) 0.664 or 66.4% b) 0.336 or 33.6%
16) a) 0.875 or 8.75% b) 0.125 or 12.5 %
17) a) 0.92 or 92% b) 0.08 or 8%
18) 0.40
19) a) $\frac{336}{720}\approx 0.47$ b) $\frac{16}{90}\approx 0.18$ c) $\frac{384}{720}\approx 0.53$
20) 0.306
21) $\frac{60}{132}\approx 0.45$
22) 0.412
23) a) 0.26 b) 0.155 c) 0.175
24) 0.583
25) a) ≈ 0.8542 b) $\frac{8000}{54872}\approx 0.1458$
26) a) 0.486 b) 0.0000000744
27) a) $\frac{40}{312}\approx 0.1282$ b) $\frac{12}{104}\approx 0.1154$
28) 0.217
29) You should switch to the other door. Switching yields a 2/3 probability of winning.
30) 0.493
31) a) $\frac{1}{2}=0.50$ b) $\frac{6}{16}=0.375$ c) $\frac{14}{16}=0.875$
32) 0.467

33) $\frac{17}{24} \approx 0.7083$

34) 0.84

35) The number of boys and girls remain about equal.

Section 2.5: Counting Methods

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me, you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger
soup + sandwich
soup + quiche
soup + fajita
soup + pizza
salad + hamburger
salad + sandwich
salad + quiche
salad + fajita
salad + pizza
breadsticks + hamburger
breadsticks + sandwich
breadsticks + quiche
breadsticks + fajita
breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

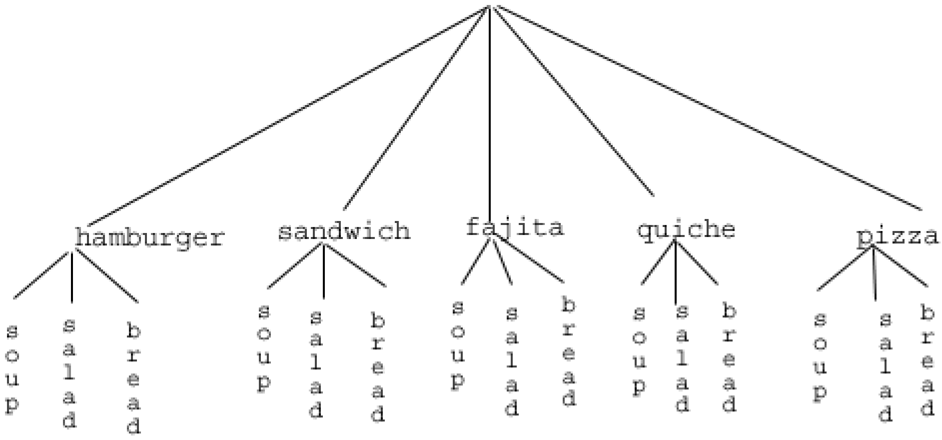
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	**sandwich**	**quiche**	**fajita**	**pizza**
soup	soup+hamburger				
salad	salad+hamburger				
bread	*etc.*				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose form instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a tree diagram, similar to those discussed in the previous section:



In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

Okay, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.

This is sometimes called the **multiplication rule for probabilities** or the **fundamental counting principle**.

Example 2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

There are three choices for an appetizer, five for the main course and two for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 4

A quiz consists of three true-or-false questions. In how many ways can a student answer the quiz?

There are three questions. Each question has two possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3, which is much more compact.

Try it Now 1

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple of examples.

Example 5

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage. We have four choices for the first letter (say we choose A), then three choices for the second (M, T and H; say we choose H), then two choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation, which shows up often in mathematics, is called the **factorial** and is notated $n!$

Factorial

$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ where $0!=1$.

Example 6

How many ways can five different door prizes be distributed among five people?

There are five choices of prize for the first person, four choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 7

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Using the basic counting rule, there are 25 choices for the person who receives the $100 certificate, 24 remaining choices for the $25 certificate and 23 choices for the $10 certificate, so there are $25 \cdot 24 \cdot 23 = 13{,}800$ ways in which the prizes can be awarded.

Example 8

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Using the Basic Counting Rule, there are eight choices for the gold medal winner, seven remaining choices for the silver, and six for the bronze, so there are 8 • 7 • 6 = 336 ways the three medals can be awarded to the eight runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

If you don't see why (n — r + 1) is the right number to use for the last factor, just think back to the first example in this section, where we calculated 25 • 24 • 23 to get 13,800. In this case n = 25 and r = 3, so n — r + 1 = 25 — 3 + 1 = 23, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$${}_nP_r = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1) \text{ or } {}_nP_r = \frac{n!}{(n-r)!}.$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*. It turns out that we can express this result more simply using factorials. Note that ${}_nP_r$ is sometimes written $P(n,r)$.

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 9

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3{,}024$ permutations.

Example 10

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43{,}680$.

Try it Now 2

How many 5 character passwords can be made using the letters A through Z

a. If repeats are allowed?

b. If no repeats are allowed?

Combinations

In the previous section we considered the situation where we chose *r* items out of *n* possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 11

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13{,}800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are 3 · 2 · 1 = 6 ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be 13,800/6 = 2300.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is **not** important*. If we are choosing *r* items out of *n* possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by $\frac{{}_nP_r}{{}_rP_r}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing *r* items out of *n* possibilities *without replacement* where the *order of selection is **not** important*.

Combinations

$${}_nC_r = \frac{{}_nP_r}{{}_rP_r}$$

We say that there are ${}_nC_r$ **combinations** of size *r* that may be selected from among *n* choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Note that ${}_nC_r$ is sometimes written $C(n,r)$.

Example 12

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is **not** important* there are ${}_{35}C_4 = \frac{35\cdot 34\cdot 33\cdot 32}{4\cdot 3\cdot 2\cdot 1}$ = 52,360 combinations.

Try it Now 3

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same, of course, would be true if the Democrats were in control.) So let's consider the problem again in a slightly more complicated form.

Example 13

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_{9} = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6{,}012{,}006$ possible ways of selecting the members of the Defense Subcommittee.

Probability Using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions.

Example 14

A four digit PIN number is selected. What is the probability that there are no repeated digits?

There are ten possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute 10 · 9 · 8 · 7, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of four digit PIN numbers with no repeated digits divided by the total number of four digit PIN numbers. This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504.$$

Example 15

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 =$ 12,271,512. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is

$$\frac{{}_{6}C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815.$$

Example 16

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12{,}271{,}512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_{6}C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_{6}C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is

$$\frac{({}_{6}C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205.$$

Try it Now 4

A multiple-choice question on an economics quiz contains ten questions with five possible answers each. Compute the probability of randomly guessing the answers and getting nine questions correct.

Example 17

Compute the probability of randomly drawing five cards from a deck and getting exactly one ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of five-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one ace and four other cards (none of them aces) from the deck. Since there are four aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one ace; since there are 48 non-aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-aces.

Putting this all together, we have $P(\text{one ace}) = \dfrac{\left({}_4C_1\right)\left({}_{48}C_4\right)}{{}_{52}C_5} = \dfrac{778320}{2598960} \approx 0.299$.

Example 18

Compute the probability of randomly drawing five cards from a deck and getting exactly two aces.

The solution is similar to the previous example, except now we are choosing two aces out of four and three non-aces out of 48; the denominator remains the same:

$$P(\text{two aces}) = \frac{\left({}_4C_2\right)\left({}_{48}C_3\right)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399.$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 5

Compute the probability of randomly drawing five cards from a deck of cards and getting three aces and two kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

> Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 19

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event:

$$\text{P(at least one)} = 1 - \text{P(none)}$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get:

$$\text{P(shared birthday)} = 1 - \text{P(no shared birthday)} = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 20

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be:

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as:

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

…which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 21

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate:

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

…which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page http://www-stat.stanford.edu/~susan/surprise/Birthday.html and, once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 6

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Try it Now Answers

1. 8 · 11 · 5 = 440 menu combinations

2. There are 26 characters.
a. 26^5 = 11,881,376 b. ${}_{26}P_5$ = 26·25·24·23·22 = 7,893,600

3. Order does not matter. ${}_{29}C_{19}$ = 20,030,010 possible subcommittees

4. There are 5^{10} = 9,765,625 different ways the exam can be answered. There are ten possible locations for the one missed question and in each of those locations there are four wrong answers, so there are 40 ways the test could be answered with one wrong answer.

   P(9 answers correct) = $\frac{40}{5^{10}}$ ≈ 0.000004096 chance

5. $P(\text{three aces and two kings}) = \frac{\left({}_4C_3\right)\left({}_4C_2\right)}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$

6. $P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{10}}{365^{10}} \approx 0.117$

Section 2.5 Exercises

1. A boy owns two pairs of pants, three shirts, eight ties, and two jackets. How many different outfits can he wear to school, if he must wear one of each item?

2. At a restaurant you can choose from three appetizers, eight entrees, and two desserts. How many different three-course meals can you have?

3. How many three-letter "words" can be made from four” letters "FGHI", if
    a. Repetition of letters is allowed.
    b. Repetition of letters is not allowed.

4. How many four-letter "words" can be made from six letters "AEBWDP", if
    a. Repetition of letters is allowed.
    b. Repetition of letters is not allowed.

5. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles if repetition of letters and numbers is allowed?

6. A computer password must be eight characters long. How many passwords are possible, if only the 26 letters of the alphabet are allowed?

7. A pianist plans to play four pieces at a recital. In how many ways can she arrange these pieces in the program?

8. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

9. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

10. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?

11. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible, if no note is repeated?

12. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the

chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?

13. In how many ways can four pizza toppings be chosen from 12 available toppings, if order doesn’t matter?

14. At a baby shower 17 guests are in attendance and five of them are randomly selected to receive a door prize. If all five prizes are identical, in how many ways can the prizes be awarded?

15. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have, if order doesn’t matter?

16. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have, if order doesn’t matter?

17. A jury pool consists of 27 people. How many different ways can eleven people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?

18. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?

19. You own 16 CDs. You want to randomly arrange five of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

20. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

21. In a lottery game, a player picks six numbers from 1 to 48. If five of the six numbers match those drawn, the player wins second prize. What is the probability of winning this prize?

22. In a lottery game, a player picks six numbers from 1 to 48. If four of the six numbers match those drawn, the player wins third prize. What is the probability of winning this prize?

23. Compute the probability that a five-card poker hand is dealt to you that contains all hearts.

24. Compute the probability that a five-card poker hand is dealt to you that contains four aces.

Section 2.5 Exercises – Answer Key

1) 96 different outfits
2) 48 different meals
3) a) 64 three-letter words b) 24 three-letter words
4) A) 4096 six-letter words b) 120 six-letter words
5) 17,576,000 different license plates
6) 2.08×10^{11} passwords
7) 24 ways
8) 9,129,120 ways
9) 35 four-person relay teams
10) 1.1876×10^{13} orders
11) 95,040 five-note themes
12) 479,001,600 tone rows
13) 495 ways
14) 6,188 ways
15) 15,890,700 choices
16) 120 ways
17) 208,606,320 ways
18) 163,350 ways
19) $\dfrac{1}{{}_{16}P_5} = \dfrac{1}{524160} \approx 0.000001908$
20) $\dfrac{{}_{14}C_{12}}{{}_{27}C_{12}} = \dfrac{91}{17,383,860} \approx 0.00000523$
21) $\dfrac{({}_{6}C_{5})({}_{42}C_{1})}{{}_{48}C_{6}} = \dfrac{252}{12271512} \approx 0.0000205$
22) $\dfrac{({}_{6}C_{4})({}_{42}C_{2})}{{}_{48}C_{6}} = \dfrac{12,915}{12,271,512} \approx 0.00105$
23) $\dfrac{{}_{13}C_{5}}{{}_{52}C_{5}} = \dfrac{1287}{2598960} \approx 0.0004952$
24) $\dfrac{({}_{4}C_{4})({}_{48}C_{1})}{{}_{52}C_{5}} = \dfrac{48}{2,598,960} \approx 0.0000185$

Section 2.6: Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it is one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 1

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then the player receives \$36 (their original \$1 + \$35). Otherwise, the player lose the \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of a \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" -\$2/38 ≈ -\$0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$. Summarizing these along with the values, we get the table below.

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get 0.9211 + (-0.9737) ≈ -0.053, which is the expected value we computed above.

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

Expected Value

Expected value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome and then adding up the products.

Try it Now 1

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example 2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match five numbers, the player wins \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all six numbers and the probability of matching five numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all six numbers,}$$

$$\frac{\left({}_6C_5\right)\left({}_{42}C_1\right)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for five numbers.}$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1-\frac{253}{12271512}=\frac{12271259}{12271512}$

The expected value is:

$$(\$999{,}999)\cdot\frac{1}{12271512}+(\$999)\cdot\frac{252}{12271512}+(-\$1)\cdot\frac{12271259}{12271512}\approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1.

In general, if the expected value of a game is negative, then it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now 2

A friend offers to play a game, in which you roll three standard six-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 3

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year. An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are:

Outcome	Probability of outcome
$100,000 - $275 = $99,725	0.00242
-$275	1 – 0.00242 = 0.99758

The expected value is ($99,725)(0.00242) + (-$275)(0.99758) = -$33.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit, because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

Try it Now Answers

1. $(\$3{,}995)\cdot\frac{1}{2000}+(-\$5)\cdot\frac{1999}{2000}$ ≈ -\$3.00

2. Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that two dice will match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

   The expected value is: $(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx$ \$0.33. Yes, it is in your advantage to play. On average, you'd win \$0.33 per play.

Section 2.6 Exercises

1. A bag contains three gold marbles, six silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select one marble from the bag; if it is gold, you win $3; if it is silver, you win $2; if it is black, you lose $1. What is your expected value if you play this game?

2. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?

3. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose $1. Find the expected value of this game.

4. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?

5. A company estimates that 0.7% of their products will fail after the original warranty period but within two years of the purchase, with a replacement cost of $350. If they offer a two year extended warranty for $48, what is the company's expected value of each warranty sold?

6. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

Section 2.6 Exercises – Answer Key

1) ≈ - $ 0.189
2) ≈ - $ 0.167
3) ≈ - $ 0.703
4) ≈ - $ 0.647
5) ≈ - $ 45.55
6) ≈ - $ 22.00

Chapter 3: Matrices

Section 3.1: Introduction to Matrices

A **matrix** is a rectangular array of numbers. It consists of rows and columns of numbers. The advantage of using a matrix to solve systems of equations is that they can hold large amounts of information, readymade for computer or graphing calculator manipulation. You may have already used matrices. A spreadsheet, such as Microsoft Excel, displays its data in a matrix format. A tax table with headings across the top and down the side is an example of a matrix. To find a particular entry, find where the corresponding row and column intersect.

However, to work with matrices we need terminology. Consider the matrix below.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Column 1} & \text{Column 2} & \text{Column 3} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \text{Row 1} \rightarrow \\ \text{Row 2} \rightarrow \end{array} & \begin{pmatrix} 4 & \quad 7 \quad & 6 \\ 5 & \quad 2 \quad & 8 \end{pmatrix} \end{array}$$

The **size**, or **dimension**, of a matrix is the number of rows and the number of columns it contains. For example, the matrix above has 2 rows and 3 columns. We call it a 2 x 3 matrix (read “2 by 3”). The size of a matrix is written on the lower right side of the matrix as illustrated below. The number of rows is listed first.

$$A = \begin{pmatrix} 4 & 7 & 6 \\ 5 & 2 & 8 \end{pmatrix}_{\underset{\substack{\text{number of rows} \\ \text{listed first}}}{\underset{\uparrow}{2}} \times \underset{\substack{\text{number of columns} \\ \text{listed last}}}{\underset{\uparrow}{3}}}$$

Matrices are named by capital letters. The name of the matrix shown above is A. This matrix has 6 entries (this is the product of the size dimensions: 2 x 3 = 6). Each entry is named by its position in the matrix (its row and column) using lower case letters. The entry 5 in the 2nd row 1st column is designated a_{21} and we write $a_{21} = 5$. The entry $a_{13} = 6$ because 6 is in the 1st row 3rd column.

A **row matrix** (or row vector) has only one row. The matrix below is a row matrix. It is a 1 x 4 matrix (meaning it has 1 row and 4 columns).

$$\begin{pmatrix} 3 & 4 & -1 & 8 \end{pmatrix} \qquad\qquad \text{row matrix}$$

A **column matrix** (or column vector) has only one column. The following matrix is a column matrix. It is a 3 x 1 matrix (meaning it has 3 rows and 1 column).

$$\begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix} \qquad \text{column matrix}$$

A **square matrix** has the same number of rows as columns. The sizes of square matrices are 2 x 2, 3 x 3, 4 x 4, etc.

Matrix Equality

Two matrices are equal if

1. They are the same size, and
2. Each pair of corresponding entries is equal.

Example 1

a) The matrices

$$\begin{pmatrix} 4 & 3 \\ -1 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & -1 \\ 3 & 6 \end{pmatrix}$$

are not equal. Though they have the same size (both are 2 x 2 square matrices) and they contain the same entries, corresponding entries (entries in the same position in each matrix) are not equal. For example, the entries 3 and -1 in the 1st row 2nd column of each matrix are not the same.

b) The matrices

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

are not equal because they are not the same size (2 x 1 on the left and 3 x 1 on the right).

c) For what values of the variables x, y, and z are the matrices equal?

$$\begin{pmatrix} 4 & x \\ y+3 & 7 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & -2 \\ 5 & z-36 \end{pmatrix}$$

Both matrices are the same size, namely 2 x 2. For corresponding pairs of entries to be equal, we need $x=-2$. We need $y+3=5$, or $y=2$, and we need $7=z-36$, or $z=43$. Therefore, the matrices are equal when $x=-2$, $y=2$, and $z=43$.

Addition of Matrices

To add two matrices, both matrices must have the same size (that is, the same number of rows and the same number of columns). The answer matrix, or the sum, has the same size as the two being added. If both matrices are the same size, add the corresponding entries.

Example 2

Add the matrices, if possible.

a) $\begin{pmatrix} 6 & -4 \\ 9 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -7 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 7 \\ 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 & 6 \\ 8 & 0 & 2 \end{pmatrix}$

Answers

a) $\begin{pmatrix} 6 & -4 \\ 9 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -7 & 0 \end{pmatrix} = \begin{pmatrix} 6+4 & -4+1 \\ 9+(-7) & 5+0 \end{pmatrix} = \begin{pmatrix} 10 & -3 \\ 2 & 5 \end{pmatrix}$

b) We cannot add the matrices A and B because they are not the same size. We say the sum A + B does not exist.

Subtraction of Matrices

To subtract two matrices,

1. Both matrices must have the same size (that is, the same number of rows and the same number of columns). The answer matrix (the difference) has the same size as the two being subtracted.
2. Subtract each entry in the matrix on the right side from its corresponding entry in the matrix on the left. (Recall, we subtract by adding the opposite of the entry on the right side).

Example 3

Subtract the matrices, if possible: $\begin{pmatrix} 4 & 7 \\ -2 & 6 \\ 0 & -5 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ 7 & 1 \\ -4 & -8 \end{pmatrix}$.

The matrices have the same size (both of 3 x 2), so we may subtract. We subtract corresponding entries.

$$\begin{pmatrix} 4 & 7 \\ -2 & 6 \\ 0 & -5 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ 7 & 1 \\ -4 & -8 \end{pmatrix} = \begin{pmatrix} 4-6 & 7-(-3) \\ -2-7 & 6-1 \\ 0-(-4) & -5-(-8) \end{pmatrix} = \begin{pmatrix} -2 & 10 \\ -9 & 5 \\ 4 & 3 \end{pmatrix}$$

Recall we subtract by adding the opposite of the second number. For example, subtracting the entries in the 3rd row 2nd column gives: $-5-(-8) = -5+(+8) = 3$.

Example 4

The entries in matrix A are the retail selling prices of men’s shirts. The entries in matrix B are the cost of each shirt to the retailer. In each matrix Row 1 is long sleeve shirts and Row 2 is short sleeve. Column1 is Arrow brand and Column 2 is Beene brand. Find the difference A – B and explain what it represents.

$$A\ \text{(selling price)} = \begin{array}{r} \\ \text{Long} \\ \text{Short} \end{array} \begin{array}{c} \begin{array}{cc} \text{Arrow} & \text{Beene} \end{array} \\ \begin{pmatrix} \$31.50 & \$26.00 \\ \$21.95 & \$16.50 \end{pmatrix} \end{array}$$

$$B\ \text{(cost)} = \begin{array}{r} \\ \text{Long} \\ \text{Short} \end{array} \begin{array}{c} \begin{array}{cc} \text{Arrow} & \text{Beene} \end{array} \\ \begin{pmatrix} \$26.25 & \$19.38 \\ \$14.00 & \$13.50 \end{pmatrix} \end{array}$$

Solution: The difference A – B represents the profit made by the retailer on each shirt.

$$A - B = \overset{\text{Selling price}}{\begin{pmatrix} \$31.50 & \$26.00 \\ \$21.95 & \$16.50 \end{pmatrix}} - \overset{\text{Cost}}{\begin{pmatrix} \$26.25 & \$19.38 \\ \$14.00 & \$13.50 \end{pmatrix}} = \overset{\text{Profit}}{\begin{pmatrix} \$5.25 & \$6.62 \\ \$7.95 & \$3.00 \end{pmatrix}}$$

For example, the profit made on the sale of one long sleeve Arrow shirt is $5.25. We can see this more clearly if we include the units for the rows and columns.

$$A - B = \begin{array}{r} \\ \text{Long Sleeve} \\ \text{Short Sleeve} \end{array} \begin{array}{c} \begin{array}{cc} \text{Arrow} & \text{Beene} \end{array} \\ \begin{pmatrix} \$5.25 & \$6.62 \\ \$7.95 & \$3.00 \end{pmatrix} \end{array}$$

Suppose we sell 60 short sleeve Beene shirts this week. From the Profit matrix we see we make a profit of $3.00 on each shirt, for a total weekly profit of $\$3 \cdot \$60 = \$180$ from this item.

Matrices are used to keep track of large amounts of data that can be arranged in a row-column format. If we sell 8 kinds of shirts and 12 brands, all of the retail selling prices and costs can be stored in two 8 x 12 matrices. By subtracting we get the profit made on each shirt. In the case of these large matrices we would use computers or graphing calculators to keep track of the larger amounts of data represented.

Try it Now 1

Given the following two matrices, find $B - A$

$$A = \begin{pmatrix} 4 & 7 \\ 0 & -8 \\ 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -2 \\ 1 & 9 \\ -5 & 0 \end{pmatrix}$$

Multiplication of Matrices

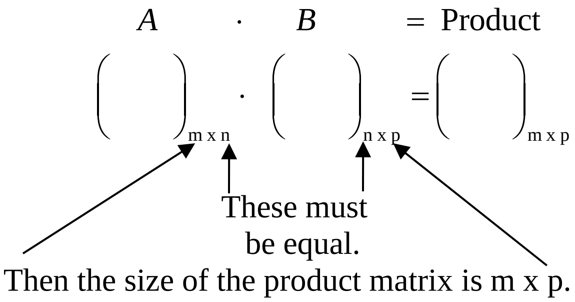
The procedure for multiplying matrices is more involved than for addition or subtraction. Understanding multiplication is essential to properly building matrices for use in applications. The answer to multiplying two matrices is called the **product**.

Multiplication of Matrices

Suppose A is an m x n matrix and B is an n x p matrix. To find the product $A \cdot B$ (refer to the figure below).

1. The number of columns in A must equal the number of rows in B. Here both are equal to n. (If they are not equal, then we cannot multiply).

2. The product is an m x p matrix. To find the entry in the ith row jth column of the product, multiply each entry in the ith row of A by the corresponding entries in the jth column of B and then add these products.



Example 5

Multiply: $\begin{pmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 6 \end{pmatrix}$

Solution: Write the size of each matrix on its lower right side. The number of columns in the first matrix equals the number of rows in the second (both are 3), so we can multiply. The answer with be a 2 x 2 matrix.

$$\begin{pmatrix}1&-2&4\\3&0&5\end{pmatrix}_{\text{2 x 3}}\cdot\begin{pmatrix}2&4\\1&2\\3&6\end{pmatrix}_{\text{3 x 2}}=\begin{pmatrix}1\cdot2+(-2)\cdot1+4\cdot3 & 1\cdot4+(-2)\cdot2+4\cdot6\\3\cdot2+0\cdot1+5\cdot3 & 3\cdot4+0\cdot2+5\cdot6\end{pmatrix}_{2x2}=\begin{pmatrix}12&24\\21&42\end{pmatrix}$$

These must
be equal.
The product will be a 2 x 2 matrix.

Each row in the first matrix is multiplied times each column in the second. For example, multiply the entries $\begin{pmatrix}1&-2&4\end{pmatrix}$ in the first row of the first matrix times the corresponding entries in the first column of the second matrix $\begin{pmatrix}2\\1\\3\end{pmatrix}$ to get the entry in the 1st row 1st column of the product. We get $1\cdot2+(-2)\cdot1+4\cdot3=12$.

When we multiply $3\cdot5$ and $5\cdot3$ we get the same answer because multiplying whole numbers is commutative. The next example shows that matrix multiplication is not commutative. If we reverse the order of the matrices, the answer may not be the same.

Example 6

Multiply the matrices in Example 1 in reverse order:

$$\begin{pmatrix}2&4\\1&2\\3&6\end{pmatrix}\cdot\begin{pmatrix}1&-2&4\\3&0&5\end{pmatrix}$$

Solution: Write the size of each matrix on its lower right side. The number of columns in the first matrix equals the number of rows in the second (both are 2), so we can multiply. The answer will be a 3 x 3 matrix.

$$\begin{pmatrix}2&4\\1&2\\3&6\end{pmatrix}_{\text{3 x 2}}\cdot\begin{pmatrix}1&-2&4\\3&0&5\end{pmatrix}_{\text{2 x 3}}=\begin{pmatrix}2\cdot1+4\cdot3 & 2\cdot(-2)+4\cdot0 & 2\cdot4+4\cdot5\\1\cdot1+2\cdot3 & 1\cdot(-2)+2\cdot0 & 1\cdot4+2\cdot5\\3\cdot1+6\cdot3 & 3\cdot(-2)+6\cdot0 & 3\cdot4+6\cdot5\end{pmatrix}_{\text{3 x 3}}=\begin{pmatrix}14&-4&28\\7&-2&14\\21&-6&42\end{pmatrix}$$

These must be equal.
The product will be a 3 x 3 matrix.

Notice that the answer to Example 2 is completely different from Example 1. Therefore, **matrix multiplication is not commutative**. Be careful to multiply matrices in the correct order. This fact will play a critical role later when solving matrix equations.

Example 7

Multiply: $\begin{pmatrix}4 & 6\\ 3 & -1\end{pmatrix}\cdot\begin{pmatrix}2 & 7\end{pmatrix}$

Solution: $\begin{pmatrix}4 & 6\\ 3 & -1\end{pmatrix}_{2\text{ x }\underset{\uparrow}{2}}\cdot\begin{pmatrix}2 & 7\end{pmatrix}_{\underset{\uparrow}{1}\text{ x }2}$ Not possible.

These are not the same.
Therefore ,we cannot multiply.

Example 8

A flu epidemic strikes a large city. Each city resident is either well, a carrier, or sick. The proportion of the population in each of these categories is given by the matrix A below and the column labels are the age groups.

$$\begin{array}{c} \\ \text{well}\\ \text{carrier}\\ \text{sick}\end{array}\begin{array}{c}\begin{array}{ccc}\text{0-18} & \text{19-45} & \text{46-up}\end{array}\\ \begin{pmatrix}.6 & .7 & .7\\ .2 & .1 & .2\\ .2 & .2 & .1\end{pmatrix}\end{array} = A \qquad \begin{array}{c} \\ \text{0-18}\\ \text{19-45}\\ \text{46-up}\end{array}\begin{array}{c}\begin{array}{cc}\text{male} & \text{female}\end{array}\\ \begin{pmatrix}300000 & 250000\\ 600000 & 650000\\ 400000 & 475000\end{pmatrix}\end{array} = B$$

a) Find the product $A\cdot B$.
b) How many sick females are there?
c) How many carriers, male or female, are in the city?

Solution:
a) *Find the product* $A\cdot B$. The product $A\cdot B$ is shown below.

$$\overset{A}{\begin{array}{c} \\ \text{well}\\ \text{carrier}\\ \text{sick}\end{array}\begin{array}{c}\begin{array}{ccc}\text{0-18} & \text{19-45} & \text{46-up}\end{array}\\ \begin{pmatrix}.6 & .7 & .7\\ .2 & .1 & .2\\ .2 & .2 & .1\end{pmatrix}_{3\text{ x }3}\end{array}} \cdot \overset{B}{\begin{array}{c} \\ \text{0-18}\\ \text{19-45}\\ \text{46-up}\end{array}\begin{array}{c}\begin{array}{cc}\text{male} & \text{female}\end{array}\\ \begin{pmatrix}300000 & 250000\\ 600000 & 650000\\ 400000 & 475000\end{pmatrix}_{3\text{ x }2}\end{array}} = \begin{array}{c} \\ \text{well}\\ \text{carrier}\\ \text{sick}\end{array}\begin{array}{c}\begin{array}{cc}\text{male} & \text{female}\end{array}\\ \begin{pmatrix}880000 & 937500\\ 200000 & 210000\\ 220000 & 227500\end{pmatrix}_{3\text{ x }2}\end{array}$$

These must be equal.

This says the number of columns in the A matrix equals the number of rows in matrix B. *This also means* that the columns labels in matrix A *must match* the row labels in the matrix B. *Then* in the product $A\cdot B$ the rows have the same labels as matrix A and the columns have the same labels as the matrix B.

Caution: Always include the row and column labels in applications. They are a helpful guide that our matrix multiplication is set up properly. If the column labels in matrix A do not match the row labels in matrix B, then consider the following.

1. You may have to interchange the rows and columns in one of your matrices.
2. Your matrix multiplication may be in the wrong order. Interchange the matrices.

Let’s look a little closer at the multiplication $A \cdot B$ above and make sense of the entries in the answer. The first row in matrix A is

$$\begin{array}{cccc} & \text{0-18} & \text{19-45} & \text{46-up} \\ \text{well} & (0.6 & 0.7 & 0.7\ ) \end{array}$$

and gives the percent of well people in each age group. For example, 60% of the 0-18 age group are well. The first column in the B matrix is

$$\begin{array}{cc} & \text{male} \\ \begin{array}{c}\text{0-18}\\ \text{19-45}\\ \text{46-up}\end{array} & \begin{pmatrix}300000\\ 600000\\ 400000\end{pmatrix} \end{array}$$

and gives the number of males in each age group. When we multiply the first row in matrix A times the first column in matrix B we get a single entry in the product matrix.

$$\begin{array}{cccc} & \text{0-18} & \text{19-45} & \text{46-up} \\ \text{well} & (0.6 & 0.7 & 0.7\ ) \end{array} \cdot \begin{array}{cc} & \text{male} \\ \begin{array}{c}\text{0-18}\\ \text{19-45}\\ \text{46-up}\end{array} & \begin{pmatrix}300000\\ 600000\\ 400000\end{pmatrix} \end{array} = \begin{array}{cc} & \text{male} \\ \text{well} & (880000) \end{array}$$

$$\begin{aligned} &= 0.6 \cdot 300000 + 0.7 \cdot 600000 + 0.7 \cdot 4000000 \\ &= 180000 + 420000 + 280000 \\ &= 880000 \end{aligned}$$

The multiplication is read as follows:

> In the 0-18 age group, 60% of the 300,000 males or 180,000 males are well.
> In the 19-45 age group, 70% of the 600,000 males or 420,000 males are well.
> In the 46-up age group, 70% of the 400,000 males or 280,000 males are well.

Adding 180,000 + 420,000 + 280,000 gives a total of 880,000 well males in our city. Similar thinking applies to the other entries in the product matrix.

b) *How many sick females are there*?

From the product matrix we see there are 227,500 sick females. Notice how easy this is to read when we include the row and column labels.

c) *How many carriers, male or female, are in the city*?

From the middle row of the product matrix there are 200,000 males carriers and 210,000 female carriers for a total of 200,000 + 210,000 = 410,000 carriers.

Try it Now 2

Given the following two matrices, find $A \cdot B$

$$A = \begin{pmatrix} 9 & 1 & -4 \\ -1 & 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -2 \\ 1 & 9 \\ -5 & 0 \end{pmatrix}$$

Try it Now Answers

1. $$B - A = \begin{pmatrix} 3 & -2 \\ 1 & 9 \\ -5 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 7 \\ 0 & -8 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 3-4 & -2-7 \\ 1-0 & 9-(-8) \\ -5-2 & 0-(-2) \end{pmatrix} = \begin{pmatrix} -1 & -9 \\ 1 & 17 \\ -7 & 2 \end{pmatrix}$$

2.

$$A \cdot B = \begin{pmatrix} 9 & 1 & -4 \\ -1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ 1 & 9 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} (9 \cdot 3) + (1 \cdot 1) + (-4 \cdot -5) & (9 \cdot -2) + (1 \cdot 9) + (-4 \cdot 0) \\ (-1 \cdot 3) + (2 \cdot 1) + (5 \cdot -5) & (-1 \cdot -2) + (2 \cdot 9) + (5 \cdot 0) \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 48 & -9 \\ -26 & 20 \end{pmatrix}$$

Section 3.1 Exercises

In problems 1-3, determine if the statement is true or false. If false, state why.

1. $$\begin{pmatrix} 4 & 5 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix}$$

2. $$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \end{pmatrix}$$

3. $$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}$$

In problems 4-7, find the size of each matrix. If a matrix is a square, row, or column matrix, identify it as such.

4. $$\begin{pmatrix} 4 & 0 & 2 \\ -1 & 7 & 1 \end{pmatrix}$$

5. $$\begin{pmatrix} 2 & 4 & 0 & -5 \end{pmatrix}$$

6. $$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

7. $$\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

In problems 8-10, find the value of the variables in each matrix.

8. $$\begin{pmatrix} 4 & y \\ 5 & z \end{pmatrix} = \begin{pmatrix} x & 2 \\ 5 & -1 \end{pmatrix}$$

9. $$\begin{pmatrix} x+6 & y-3 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ z+3 & 1 \end{pmatrix}$$

10. $$\begin{pmatrix} -3 \\ y-2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5-x \\ -8 \\ z+9 \end{pmatrix}$$

In problems 11-14, perform the indicated operations. If not possible, state why.

11. $$\begin{pmatrix} 5 & -3 & 8 \\ 0 & 4 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 7 & 4 \\ -5 & -9 & -2 \end{pmatrix}$$

12. $$\begin{pmatrix} 5 & -3 & 8 \\ 0 & 4 & -6 \end{pmatrix} - \begin{pmatrix} 1 & 7 & 4 \\ -5 & -9 & -2 \end{pmatrix}$$

13. $$\begin{pmatrix} 5 & -2 & 0 \\ 9 & 0 & -7 \\ 4 & 8 & 1 \end{pmatrix} - \begin{pmatrix} 8 & -7 & 4 \\ 3 & -1 & -5 \\ -3 & 8 & -1 \end{pmatrix}$$

14. $$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

In problems 15-20, the sizes of two matrices A and B are given. Find the size of the product $A \cdot B$ and the product $B \cdot A$. If the product does not exist, say so.

15. A is 2 x 2 and B is 2 x 3
16. A is 2 x 4 and B is 3 x 2
17. A is 3 x 2 and B is 2 x 3
18. A is 4 x 1 and B is 1 x 4
19. A is 2 x 6 and B is 4 x 6
20. A is 3 x 5 and B is 2 x 5

In problems 21-32, find the product of the matrices. If the product does not exist, say so.

21. $\begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 6 & 0 \\ 4 & 3 \end{pmatrix}$

22. $\begin{pmatrix} 6 & -3 \\ 7 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}$

23. $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 & -1 \end{pmatrix}$

24. $\begin{pmatrix} 4 & 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$

25. $\begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

26. $\begin{pmatrix} 2 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix}$

27. $\begin{pmatrix} 5 & -3 & 8 \\ 2 & 4 & -6 \end{pmatrix} \cdot \begin{pmatrix} 2 & -5 \\ 3 & 0 \\ -1 & 4 \end{pmatrix}$

28. $\begin{pmatrix} 3 & 8 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 4 \\ 5 & -4 & 2 \end{pmatrix}$

29. $\begin{pmatrix} 2 & -3 \\ 5 & 4 \end{pmatrix}^2$

30. $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 2 \\ -3 & 1 & -1 \end{pmatrix}^2$

31. $\begin{pmatrix} 5 & -2 & 0 \\ 9 & 0 & -7 \\ 4 & 8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

32. $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & -1 \\ 0 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 & 1 \\ 4 & -2 & 0 \\ 3 & 5 & 6 \end{pmatrix}$

33. **Fast Food Chain**: A small fast food chain with three stores sells three dinners: chicken, steak, and fish. Matrix A gives the wholesale cost of each dinner. Matrix B gives the retail selling price of each dinner. Matrix C gives the number of each dinner sold in the three stores in April. Matrix D give the number of each dinner sold in May.

Wholesale Cost

$$A = \begin{array}{c} \text{chicken} & \text{steak} & \text{fish} \\ \left(\$4\right. & \$7 & \left.\$5\right) \end{array}$$

Retail Selling Price

$$B = \begin{array}{c} \text{chicken} & \text{steak} & \text{fish} \\ \left(\$6\right. & \$10 & \left.\$8.50\right) \end{array}$$

April Sales

$$C = \begin{array}{c|ccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{chicken} & 60 & 70 & 50 \\ \text{steak} & 80 & 60 & 90 \\ \text{fish} & 50 & 50 & 40 \end{array}$$

May Sales

$$D = \begin{array}{c|ccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{chicken} & 70 & 80 & 60 \\ \text{steak} & 60 & 75 & 80 \\ \text{fish} & 40 & 60 & 55 \end{array}$$

Perform each operation and then explain the meaning of the answer matrix to the business. If the operation makes no practical sense, explain why.

a) $B-A$
b) $A-B$
c) $C+D$
d) $D-C$
e) $A\cdot C$
f) $A\cdot D$
g) $B\cdot C$
h) $B\cdot D$
i) $(B-A)\cdot C$
j) $(B-A)\cdot D$
k) $(B-A)\cdot(C+D)$
l) $C\cdot D$

34. **Manufacturing**: A company makes DVD players and Blu-ray players. Each DVD player requires 3 hours for assembly and ¼ hour for packaging. Each Blu-ray player requires 2 hours for assembly and ½ hour for packaging.

a) Write a matrix P containing the times for assembly and packaging of DVD players and Blu-ray players. Label each row with equipment and each column with a process.
b) The company receives an order from a retailer for 50 DVD players and 35 Blu-ray players. Build a row matrix R for this order. Label each row and column.
c) Which product or makes sense. Find the product that makes sense (label the row and columns). Explain why the other product does not make sense.
d) The product that makes sense gives us the total assembly time and the total packaging time to fill the order. What is the total assembly time? Packaging time?

35. **Contracting**: A building contractor builds four kinds of houses: cottage style, rancher, split level, and colonial. Each is built as either a 3-bedroom 1 ½ bath home or 4-bedroom 2 ½ bath home. The builder plans to construct a total of 160 homes in a new subdivision. Matrix B gives the number of each kind of home planned for the subdivision.

$$B = \begin{array}{r|cc|} & \text{3-BR} & \text{4-BR} \\ \text{cottage} & 10 & 10 \\ \text{rancher} & 10 & 20 \\ \text{split level} & 20 & 30 \\ \text{colonial} & 20 & 40 \end{array}$$

The amount of interior materials (ordered in units) needed for the construction of each size home is given in Matrix M. Note that lumber is in units of 50 board feet, drywall is in 50 sheet units, flooring per 100 sq. ft., electrical per 10 outlets, and plumbing per fixture.

$$M = \begin{array}{r|ccccc|} & \text{lumber} & \text{drywall} & \text{flooring} & \text{elec.} & \text{plumb.} \\ \text{3-BR} & 10 & 4 & 20 & 32 & 8 \\ \text{4-BR} & 12 & 6 & 25 & 40 & 12 \end{array}$$

Matrix C gives the cost in dollars of one unit of each material.

$$C = \begin{array}{r|c|} & \text{cost per unit} \\ \text{lumber} & 300 \\ \text{drywall} & 150 \\ \text{flooring} & 75 \\ \text{electrical} & 350 \\ \text{plumbing} & 150 \end{array}$$

a) Use matrix multiplication to find the total amounts of material needed to construct each of the four models of home. State which matrices you multiplied and find the product.

b) How many entries are in this matrix? Interpret the meaning of the entry in the 2nd row 3rd column.

c) Use matrix multiplication again to find the total cost of interior materials for each of the four models. State which matrices you multiplied and find the product. What is the total cost of materials to build all of the cottages? All of the ranchers? All of the split levels? All of the colonials?

d) What is the total cost for all of the interior materials used in the construction of the 160 homes in this subdivision?

36. **Analyzing Cost**: The Sweetum Candy Company makes three types of chocolate candy: Chunky Cherry, Mango Mocha, and Juicy Java. The company makes its products in Los Angeles, Tijuana, and Havana using two main ingredients: sugar and chocolate.

a) Each kilogram of Chunky Cherry requires 0.6 kg of sugar and 0.2 kg of chocolate. Each kilogram of Mango Mocha requires 0.4 kg of sugar and 0.3 kg of chocolate. Each kilogram of Juicy Java requires 0.3 kg of sugar and 0.5 kg of chocolate. Put this information into a 2 x 3 matrix. Label the rows and columns.

b) The cost of 1 kg sugar is $4 in Los Angeles, $3.25 in Tijuana, and $2 in Havana. The cost of 1 kg of chocolate is $3.50 in Los Angeles, $3.25 in Tijuana, and $4 in Havana. Put this information into a matrix in such a way that when you multiply it with your matrix in part a, you get a matrix representing the ingredient cost of making each type of candy in each city.

c) Multiply the matrices in parts a and b. Label each row and column of the answer matrix.

d) From the answer in part c, what is the combined sugar plus chocolate cost to produce 1 kg of Mango Mocha in Havana?

e) Sweetum Candy has a rush order for 200 kg of Chunky Cherry, 350 kg of Mango Mocha, and 800 kg of Juicy Java. It decides to select one of its factories to fill the entire order. Use matrix multiplication to determine in which city the total sugar plus chocolate cost to produce the order is the least.

Section 3.1 Exercises – Answer Key

1. False; not all corresponding entries are equal.
2. False; the matrices are not the same size.
3. False; the matrices are not the same size.
4. 2 x 3
5. 1 x 4; row matrix
6. 2 x 1; column matrix
7. 3 x 3; square matrix
8. $x = 4, y = 2, z = -1$
9. $x = -2, y = 5, z = 4$
10. $x = 8, y = -6, z = 3$
11. $\begin{pmatrix} 6 & 4 & 12 \\ -5 & -5 & -8 \end{pmatrix}$
12. $\begin{pmatrix} 4 & -10 & 4 \\ 5 & 13 & -4 \end{pmatrix}$
13. $\begin{pmatrix} -3 & 5 & -4 \\ 6 & 1 & -2 \\ 7 & 0 & 2 \end{pmatrix}$
14. Not possible; the matrices are not the same size.
15. $A \cdot B$ is 2 x 3; $B \cdot A$ does not exist because $3 \neq 2$
16. $A \cdot B$ does not exist because $4 \neq 3$; $B \cdot A$ is 3 x 4 does not exist because $3 \neq 2$

17. $A \cdot B$ is 3 x 3; $B \cdot A$ is 2 x 2
18. $A \cdot B$ is 4 x 4; $B \cdot A$ is 1 x 1
19. $A \cdot B$ does not exist because $6 \neq 4$; $B \cdot A$ does not exist because $6 \neq 2$
20. $A \cdot B$ does not exist because $5 \neq 2$; $B \cdot A$ does not exist because $5 \neq 3$
21. $\begin{pmatrix} 36 & 9 \\ 8 & 15 \end{pmatrix}$
22. $\begin{pmatrix} 33 & -3 \\ 22 & 24 \end{pmatrix}$
23. $\begin{pmatrix} -12 & -15 & 3 \\ 8 & 10 & -2 \\ 24 & 30 & -6 \end{pmatrix}$
24. (-8)
25. $\begin{pmatrix} 32 \\ -22 \end{pmatrix}$
26. Product does not exist because $1 \neq 2$
27. $\begin{pmatrix} -7 & 7 \\ 22 & -34 \end{pmatrix}$

28. $\begin{pmatrix} 43 & -23 & 28 \\ 23 & -26 & 2 \end{pmatrix}$

29. $\begin{pmatrix} -11 & -18 \\ 30 & 1 \end{pmatrix}$

30. $\begin{pmatrix} -5 & 2 & 0 \\ -8 & 3 & -2 \\ -1 & 0 & -3 \end{pmatrix}$

31. $\begin{pmatrix} 12 \\ -3 \\ 3 \end{pmatrix}$

32. $\begin{pmatrix} 26 & 11 & 25 \\ 17 & -19 & -4 \\ 23 & 21 & 30 \end{pmatrix}$

33.

a. The answer $\begin{pmatrix} \$2 & \$3 & \$3.50 \end{pmatrix}$ is the profit for each item.

b. The answer $\begin{pmatrix} -\$2 & -\$3 & -\$3.50 \end{pmatrix}$ has no practical meaning.

c. $$\begin{array}{c c} & \begin{array}{ccc} \text{Store1} & \text{Store2} & \text{Store3} \end{array} \\ \begin{array}{c} \text{chicken} \\ \text{steak} \\ \text{fish} \end{array} & \begin{pmatrix} 130 & 150 & 110 \\ 140 & 135 & 170 \\ 90 & 110 & 95 \end{pmatrix} \end{array}$$

This is April sales plus May sales. An answer such as $a_{1,1} = 130$ means that the total chicken dinners sold in store 1 for April and May was 130.

d. $$\begin{array}{c c} & \begin{array}{ccc} \text{Store1} & \text{Store2} & \text{Store3} \end{array} \\ \begin{array}{c} \text{chicken} \\ \text{steak} \\ \text{fish} \end{array} & \begin{pmatrix} 10 & 10 & 10 \\ -20 & 15 & -10 \\ -10 & 10 & 15 \end{pmatrix} \end{array}$$

This is May sales minus April sales. A positive answer such as $a_{1,1} = 10$ means there were 10 more chicken dinners sold by store 1 in May than April. A negative entry such as $a_{2,1} = 20$ means there were 20 fewer steak dinners sold by store 1 in May compared to April.

e. $$\begin{array}{c c} & \begin{array}{ccc} \text{Store1} & \text{Store2} & \text{Store3} \end{array} \\ \text{cost} & \begin{pmatrix} \$1050 & \$950 & \$1030 \end{pmatrix} \end{array}$$

The cost for all dinners for each store in April. For example, in April the cost for store 1 for all dinners was \$1050.

f. $$\begin{array}{c c} & \begin{array}{ccc} \text{Store1} & \text{Store2} & \text{Store3} \end{array} \\ \text{cost} & \begin{pmatrix} \$900 & \$1145 & \$1075 \end{pmatrix} \end{array}$$

The cost for all dinners for each store in May. For example, in May the cost for store 1 for all dinners was $900.

g. $$\begin{array}{cccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{sales} & (\$1585 & \$1445 & \$1540) \end{array}$$

Total sales for all dinners in April for each store.

h. $$\begin{array}{cccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{sales} & (\$1360 & \$1740 & \$1627.50) \end{array}$$

Total sales for all dinners in May for each store.

i. $$\begin{array}{cccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{sales} & (\$1535 & \$495 & \$510) \end{array}$$

The profit for all dinners in April for each store.

j. $$\begin{array}{cccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{sales} & (\$460 & \$595 & \$552.50) \end{array}$$

The profit for all dinners in May for each store.

k. $$\begin{array}{cccc} & \text{Store1} & \text{Store2} & \text{Store3} \\ \text{sales} & (\$995 & \$1090 & \$1062.50) \end{array}$$

The total profit for all dinners in April and May for each store

l. Multiplying April sales and May sales, while you get an answer, has no practical meaning. Note that the column labels in matrix C do not match the row labels in matrix D, as they should for a sensible answer.

34.

a. $$P = \begin{array}{ccc} & \text{assembly} & \text{packaging} \\ \text{DVD} & 3 & 1/4 \\ \text{Blu-ray} & 2 & 1/2 \end{array}$$

b. $$R = \begin{array}{ccc} & \text{DVD} & \text{Blu-ray} \\ \text{retail order} & (50 & 35) \end{array}$$

c. $$R \cdot P = \begin{array}{ccc} & \text{assembly} & \text{packaging} \\ \text{hours} & (220 & 30) \end{array}$$

The product $P \cdot R$ is not possible.

d. The total assembly time is 220 hours and the total packaging time is 30 hours.

35.

a. $$B \cdot M = \begin{array}{cccccc} & \text{lumber} & \text{drywall} & \text{flooring} & \text{elec.} & \text{plumb.} \\ \text{cottage} & 220 & 100 & 450 & 720 & 200 \\ \text{rancher} & 340 & 160 & 700 & 1120 & 320 \\ \text{split level} & 560 & 260 & 1150 & 1840 & 520 \\ \text{colonial} & 680 & 320 & 1400 & 2240 & 640 \end{array}$$

b. There are 20 entries in the matrix. The entry in the 2nd row 3rd column, 700, means that the total amount of flooring needed to construct the rancher homes is 70000 sq. ft.

c. $$(B\cdot M)\cdot C = \begin{array}{rc} \text{cottage} & 396750 \\ \text{rancher} & 618500 \\ \text{split level} & 1015250 \\ \text{colonial} & 1237000 \end{array}$$

The total cost to build all of the cottages is $396,750. The total cost to build all of the ranchers is $618,500.The total cost to build all of the split levels is $1,015,250. The total cost to build all of the colonials is $1,237,000.

d. The total cost of all interior materials used in the construction of the homes is $3,267,500.

36.

a. $$\begin{array}{rccc} & \text{Chunky Cherry} & \text{Mango Mocha} & \text{Juicy Java} \\ \text{sugar} & 0.6 & 0.4 & 0.3 \\ \text{chocolate} & 0.2 & 0.3 & 0.5 \end{array}$$

b. $$\begin{array}{rcc} & \text{sugar} & \text{chocolate} \\ \text{Los Angeles} & 4.00 & 3.50 \\ \text{Tijuana} & 3.25 & 3.25 \\ \text{Havana} & 4.00 & 4.00 \end{array}$$

c. $$\begin{array}{rccc} & \text{Chunky Cherry} & \text{Mango Mocha} & \text{Juicy Java} \\ \text{Los Angeles} & 3.1 & 2.65 & 2.95 \\ \text{Tijuana} & 2.6 & 2.275 & 2.6 \\ \text{Havana} & 3.2 & 2.8 & 3.2 \end{array}$$

d. The combined sugar plus chocolate cost to produce 1 kg of Mango Mocha in Havana is $2.80.

e. $$\begin{array}{rccc} & \text{Chunky Cherry} & \text{Mango Mocha} & \text{Juicy Java} \\ \text{Los Angeles} & 3.1 & 2.65 & 2.95 \\ \text{Tijuana} & 2.6 & 2.275 & 2.6 \\ \text{Havana} & 3.2 & 2.8 & 3.2 \end{array} \cdot \begin{array}{rc} & \text{kg} \\ \text{Chunky Cherry} & 200 \\ \text{Mango Mocha} & 350 \\ \text{Juicy Java} & 800 \end{array} = \begin{array}{rc} & \text{cost} \\ \text{Los Angeles} & 3907.50 \\ \text{Tijuana} & 3396.25 \\ \text{Havana} & 4180.00 \end{array}$$

The rush order should be completed by the factory in Tijuana.

Section 3.2: Gaussian Elimination Method

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 A.D. in China, describes a formula very similar to the Gaussian Elimination Method and is very similar to the Addition Method.

The Gaussian Elimination Method (GEM) method allows us to solve systems of equations using matrices. We illustrate the process by solving a system of two equations in two unknowns.

The 3 Elementary Row Operations for Matrices

1. Interchange (or swap) any two rows.
2. Multiply (or divide) one row by a nonzero number, add the result to another row. (This does not change the original row.) This operation is often used to create 0 entries.
3. Multiply (or divide) a row by a nonzero number. This is often used to create 1 entries.

Example 1

Solve using GEM: $\begin{aligned} x+3y&=7 \\ 3x+4y&=11 \end{aligned}$

The three elementary row operations are actually operations we use to solve a system of equations by hand. However, as will become clear we do not need to x and y variables to be present when we solve a system. When we add, subtract, multiply, and divide in an equation, it is the numbers that we add, subtract, multiply, and divide. The variables x and y tag along for the ride so we know which number is associated with which variable.

Initial Augmented Matrix
$$\begin{array}{c} x \quad\;\; y \quad\;\; \\ \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right) \end{array}$$

Our goal is to use the 3 elementary row operations to change this into **diagonal form**
$$\begin{array}{c} x \qquad\quad y \qquad\quad \\ \left(\begin{array}{cc|c} \underset{\text{First}}{1} & \underset{\text{Fourth}}{0} & a \\ \underset{\text{Second}}{0} & \underset{\text{Third}}{1} & b \end{array}\right) \end{array}$$

The **initial augmented matrix** on the upper left contains the coefficients and constants from our system of equations. The first column is x-coefficients, the second column is y-coefficients, and the third column is the constants, which is separated from the coefficients by a vertical line. Each row is an equation. For example, the first row is the first equation. We read it $1\cdot x+3\cdot y=7$ or $x+3y=7$.

Our job is to use the three row operations to convert the initial augmented matrix on the left into **diagonal form**, shown in in the matrix on the right. In diagonal form, 1s are the entries on the diagonal on the left side of the vertical line. All of the other entries are 0s. The solutions are $x = a$ and $y = b$, and appear on the right side of the vertical line. We verify this by converting each row into an equation. For example, converting the first row into an equation gives $1 \cdot x + 0 \cdot y = a$, or $x = a$.

The matrix on the upper right shows the procedure for reducing the matrix on the left into diagonal form. (This is not the only procedure, but it is often the fastest.) We read this procedure as follows. Use the three row operations to:

> First, convert the upper left entry in the augmented matrix into a 1.
> Second, use this 1 to change the lower left entry into a 0.
> Etc. The complete solution is below.

Solution:

$$\begin{array}{rcl}
 & \begin{array}{cc} x & y \end{array} & \\
\begin{array}{r}\text{initial}\\ \text{matrix}\end{array} & \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right) & \begin{array}{l} \leftarrow \text{ row 1 or R1} \\ \leftarrow \text{ row 2 or R2} \end{array}
\end{array}$$

First check to see that the upper left entry is a 1.

Next, use row operation #2 to change the lower left entry into a 0. The rest of the solution follows.

$$\xrightarrow[\text{add to R2}]{-3\times \text{R1}} \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{\text{R2} \div -5} \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[\text{add to R1}]{-3\times \text{R2}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

This matrix is now in → diagonal form. Write the solution by reading each row as an equation.

$$\begin{array}{lcl} 1x + 0y = 1 & \text{or} & x = 1 \\ 0x + 1y = 2 & \text{or} & y = 2 \end{array}$$

The solution $x=1$ and $y=2$ can be written as the ordered pair (1, 2). Check the solution by substituting $x=1$ and $y=2$ into each of the original equations:

$x+3y=1\cdot(1)+3\cdot(2)=1+6=7$ Checks
$3x+4y=3\cdot(1)+4(2)=3+8=11$ Checks

Example 2

Use GEM to solve the system of equations: $$\begin{aligned}2x+3y+z&=9\\4x-y+3z&=-1\\6x+2y-4z&=-8\end{aligned}.$$

Discussion:

$$\begin{array}{cc} & \underline{\text{Original Augmented Matrix}} \\ & \begin{array}{c|ccc|c} & x & y & z & \\ \text{Row 1} & 2 & 3 & 1 & 9 \\ \text{Row 2} & 4 & -1 & 3 & -1 \\ \text{Row 3} & 6 & 2 & -4 & -8 \end{array} \end{array} \quad \begin{array}{c}\text{change}\\ \rightarrow \\ \text{into}\end{array} \quad \begin{array}{c} \underline{\text{Final Goal: Diagonal Form}} \\ \begin{array}{ccc|c} x & y & z & \\ 1^{st}=1 & 5^{th}=0 & 9^{th}=0 & \text{Ans } x \\ 2^{nd}=0 & 4^{th}=1 & 8^{th}=0 & \text{Ans } y \\ 3^{rd}=0 & 6^{th}=0 & 7^{th}=1 & \text{Ans } z \end{array} \end{array}$$

The object is to apply the three row operations to the augmented matrix on the left to get the diagonal form on the right. When in diagonal form, 1s are on the diagonal l, 0s are elsewhere, and the answers are on the right side of the line. There are nine steps to this procedure, as shown above right.

First, change the 2 on the left into a 1.
Second, change the 4 into a 0.
Third, change the 6 into 0. Etc.

Solution:

The first step is to change the 2 in the upper left into a 1. **Generally we use row operation #3 to change an entry into a 1**. Divide each entry in Row 1 by 2. We designate this $\text{R}1\div 2$.

$$\begin{array}{c|ccc|c} & x & y & z & \\ \text{Row 1} & 2 & 3 & 1 & 9 \\ \text{Row 2} & 4 & -1 & 3 & -1 \\ \text{Row 3} & 6 & 2 & -4 & -8 \end{array} \quad \xrightarrow{\text{R}1\div 2} \quad \begin{array}{ccc|c} x & y & z & \\ 1 & 1.5 & 0.5 & 4.5 \\ 4 & -1 & 3 & -1 \\ 6 & 2 & -4 & -8 \end{array}$$

Note only Row 1 changes. The second step is to use the 1 in the upper left corner to change 4 into 0. **Generally we use row operation #2 to change an entry into 0**. Multiply Row 1 by -4 and add the result to Row 2. We designate this **-4× R1 add to R2**.

$$\begin{array}{ccc} & \begin{array}{cccc} x & y & z & \end{array} & \\ \begin{array}{c} -4\times\text{R1} \\ \rightarrow \\ \text{add to R2} \end{array} & \left(\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 4.5 \\ 4 & -1 & 3 & -1 \\ 6 & 2 & -4 & -8 \end{array}\right) & \begin{array}{c} \text{Row 1 does not change!} \\ \text{Row 2 changes because we} \\ \text{added } -4 \text{ times Row 1 to it.} \end{array} \end{array}$$

Now use the 1 on the upper left to change the 6 into a 0.

$$\begin{array}{ccc} & \begin{array}{cccc} x & y & z & \end{array} & \\ \begin{array}{c} -6\times\text{R1} \\ \rightarrow \\ \text{add to R3} \end{array} & \left(\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 4.5 \\ 0 & -7 & 1 & -19 \\ 0 & -7 & -7 & -35 \end{array}\right) & \begin{array}{c} \text{Row 1 does not change!} \\ \text{Row 3 changes because we} \\ \text{added } -6 \text{ times Row 1 to it.} \end{array} \end{array}$$

The leftmost column is now finished. Begin working on the second column. The goal tells us the 4th step is to change -7 in the middle of the second column into a 1. We do this by dividing Row 2 by -7. But, you cry, this gives us fractions! But fret not. Notice if we divide Row 3 by -7 we avoid fractions. This is where the swap from row operation #1 comes in handy! Swap Rows 2 and 3 and then divide Row 2 by -7. (The point here is clear: you have flexibility when applying the row operations.)

$$\begin{array}{cccc} & \begin{array}{cccc} x & y & z & \end{array} & & \begin{array}{cccc} x & y & z & \end{array} \\ \begin{array}{c} \text{Swap R2} \\ \rightarrow \\ \text{with R3} \end{array} & \left(\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 4.5 \\ 0 & -7 & -7 & -35 \\ 0 & -7 & 1 & -19 \end{array}\right) & \begin{array}{c} \text{R2}\div(-7) \\ \rightarrow \\ \end{array} & \left(\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 4.5 \\ 0 & 1 & 1 & 5 \\ 0 & -7 & 1 & -19 \end{array}\right) \end{array}$$

Now use the 1 in the second column to make the -7 into a 0 and to make 1.5 into a 0. We can perform both operations in the same step to save space. Note, Row 2 will not change. We use it to change Rows 1 and 3.

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & \end{array} \\ \begin{array}{c} -1.5\times\text{R2 add to R1} \\ \rightarrow \\ 7\times\text{R2 add to R3} \\ \rightarrow \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 8 & 16 \end{array}\right) \end{array}$$

Column 2 is now finished. Here is where we see the beauty of this procedure. While we made changes in the second column, the first column did not change. We did not undo the previous work we finished. Multiplying 0 by -1.5 or 7 in the middle of the first column gives 0 and adding 0 to the 1 above or the 0 below does not change it.

The 7th step is to change the 8 into a 1.

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} \text{R3} \div 8 \\ \rightarrow \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \end{array}$$

Finally, use the 1 in Row 3 to change the 1 and -1 above it into 0s.

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1\times \text{R3 add to R1} \\ \rightarrow \\ -1\times \text{R3 add to R2} \\ \rightarrow \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \end{array}$$

Solution: read each row as an equation:

$$\begin{array}{lcl} 1x + 0y + 0z = -1 & \text{or} & x = -1 \\ 0x + 1y + 0z = 3 & \text{or} & y = 3 \\ 0x + 0y + 1z = 2 & \text{or} & z = 2 \end{array}$$

The solution $x = 1$, $y = 3$, and $z = 2$ can be written as the ordered triple (-1, 3, 2). To check, substitute these values into each of the original equations (all three must check!).

$$\begin{array}{ll} 2x + 3y + z = 2(-1) + 3(3) + (2) = -2 + 9 + 2 = 9 & \text{Checks} \\ 4x - y + 3z = 4(-1) - (3) + 3(2) = -4 + (-3) + 6 = -1 & \text{Checks} \\ 6x + 2y - 4z = 6(-1) + 2(3) - 4(2) = -6 + 6 + (-8) = -8 & \text{Checks} \end{array}$$

World View Note: The French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables x, y, and z.

Example 3

Use GEM to solve the system of equations: $\begin{array}{r} x - 3y = 5 \\ -2x + 6y = -4 \end{array}$.

Solution: Write the augmented matrix for the system, then solve.

$$\left(\begin{array}{cc|c} 1 & -3 & 5 \\ -2 & 6 & -4 \end{array}\right) \begin{array}{c} 2\times \text{Row 1} \\ \rightarrow \\ \text{add to Row 2} \end{array} \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 0 & 6 \end{array}\right) \begin{array}{c} \text{No} \\ \rightarrow \\ \text{Solution!} \end{array} \begin{array}{c} 1x - 3y = 5 \\ 0x + 0y = 6 \quad \text{or} \quad 0 = 6 \quad \text{False!} \end{array}$$

The bottom row of the second matrix, to the left of the vertical line, has all 0 entries. Therefore, it is impossible to write this matrix in diagonal form (we cannot get a 1 below the -3 without changing the 0 below the 1). So we stop. When we translate each row of the matrix into an equation we notice something else. The second row gives us a false equation 0 = 6. This tells us the system of equations we started with has no solution.

There is a sensible geometric interpretation to this answer. In Example 1, the graph of each equation is a straight line. The solution of the system, (4, -2), is the point where the two lines intersect. In Example 3, there is no solution, so there is no point where the two lines intersect. Therefore the lines must be parallel! Since parallel lines do no intersect there is no solution to the system in this example.

Example 4

Use GEM to solve the system of equations: $\begin{matrix} x-3y=5 \\ -2x+6y=-4 \end{matrix}$.

Solution: Write the augmented matrix for the system, then solve.

$$\left(\begin{array}{cc|c} 1 & -2 & -3 \\ -4 & 8 & 12 \end{array}\right) \xrightarrow[\text{add to Row 2}]{4\times\text{Row 1}} \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{Solutions!}]{\text{Infinite}} \begin{matrix} 1x-2y=-3 \\ 0x+0y=0 \quad \text{or} \quad 0=0 \quad \text{True!} \end{matrix}$$

The bottom row of the matrix has all 0 entries. Therefore, it is impossible to write this matrix in diagonal form. So we stop. A row of 0s indicates infinitely many solutions. When we translate each row of the matrix into an equation, we notice that the second row gives us the equation 0 = 0, a true statement. This tells us that the two original equations are the same (in fact, if you multiply the first equation by -4 you get the second equation). As a result, there are an infinite number of solutions. Geometrically, since the equations are the same, the graph of the second equation is the same as the graph of the first. So every point (solution) on the graph of the second equation is also a point (solution) on the graph of the first. Since there are infinitely many such points, there are infinitely many solutions to the system.

When we reduce a matrix using the GEM method and a row becomes all 0s, this often indicates infinitely many solutions to the system. We write all of the solutions by defining a parameter. For instance, in Example 4, we get the equation $x-2y=-3$ from the final matrix. Suppose we let x be the parameter. Solve the equation for the other variable y. We get $y=\frac{x}{2}+\frac{3}{2}$. We can then write all of the solutions to the system in Example 4 as an ordered pair (x, y), in the form $(x,\frac{x}{2}+\frac{3}{2})$. Note, we just substituted $\frac{x}{2}+\frac{3}{2}$ for y. To find a solution just pick a value for x. If we pick $x=5$, then $y=\frac{x}{2}+\frac{3}{2}=y=\frac{5}{2}+\frac{3}{2}=\frac{8}{2}=4$. The ordered pair (5, 4) is a solution and satisfies both of the original equations. If we pick $x=-7$, then $y=-\frac{7}{2}+\frac{3}{2}=-\frac{4}{2}=-2$ and we have another solution (-7, -2). It also satisfies both original equations. Pick any value for x, substitute to find y and we get another solution. In this way we can generate solutions.

World View Note: The Babylonians were the first to work with systems of equations in two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables. Then, around 300 A.D., the Greeks developed methods for solving systems of equations with any number of unknowns.

Example 5

Use GEM to solve the following system of equations: $\begin{aligned} 2x &= 4+y \\ x-3y-1 &= z-5y \\ 107 &= 2y+5z+3x \end{aligned}$.

Solution:

To solve a system of equations using matrices, each equation must have all letters on the left side of the equals sign and all constants on the right side. This is called s**tandard form** for the system of equations. We write each of the original three equations in standard form as follows.

$2x = 4+y \rightarrow 2x-y=4$ Subtract y form each side to get all letters on the left.

$x-3y-1=z-5y \rightarrow x+2y-z=1$ Add 1 to each side to get the constant on the right. Add 5y and subtract z from each side to get all letters on the left side.

$107=2y+5z+3x \rightarrow 3x+2y+5z=107$ Interchange the left and right sides of the equation. Put the letters in alphabetical order.

Now write the augmented matrix for the system using the equations written in standard form and solve.

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ 1 & 2 & -1 & 1 \\ 3 & 2 & 5 & 107 \end{array}\right) \xrightarrow[\text{left to the student}]{\text{The steps are}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 16 \end{array}\right) \xrightarrow{\text{Solution}} \begin{aligned} x&=5 \\ y&=6 \\ z&=16 \end{aligned}$$

The solution is written as the ordered triple (5, 6, 16). Check it. Keep in mind the solution must check in all three of the original equations (not in the standard form equations, but in the original three equations! Can you explain why?).

Try it Now 1

Use GEM to solve the system of equations: $\begin{aligned} x+y&=15 \\ 2x-y&=12 \end{aligned}$.

Example 6

Frank’s Furniture Company has the following labor restrictions: 1,710 machine hours are available each week in the cutting department, 1,340 hours in the assembly department, and 1,880 hours in the finishing department. Manufacturing a chair requires 0.2 hours of cutting, 0.3 hours of assembly, and 0.1 hours of finishing. Each cabinet requires 0.5 hours of cutting, 0.4 hours of assembly, and 0.6 hours of finishing. Each buffet requires 0.3 hours of cutting, 0.1 hours of assembly, and 0.4 hours of finishing. How many chairs, cabinets, and buffets can the company produce in order to use all of the available labor?

First, summarize all of the given information in a table.

Item	Cutting	Assembly	Finishing	Number to Make
Chair	0.2 hours	0.3 hours	0.1 hours	x chairs
Cabinet	0.5 hours	0.4 hours	0.6 hours	y cabinets
Buffet	0.3 hours	0.1 hours	0.4 hours	z buffets
Available Labor	1,710 hours	1,340 hours	1,880 hours	

Second, build equations from the table information. Each operation (the cutting, assembling, and finishing columns) in the manufacturing process gives us an equation.

Cutting equation: $0.2x+0.5y+0.3z=1{,}710$

Assembly equation: $0.3x+0.4y+0.1z=1{,}340$

Finishing equation: $0.1x+0.6y+0.4z=1{,}880$

Each equation reads like a sentence and each term has meaning. For example, the term $0.2x$ in the cutting equation says 0.2 hours spent cutting material for each chair times the number of chairs x gives a total of $0.2x$ hours of labor spent on cutting for chairs.

Third, build an augmented matrix for this system of equations and solve using the GEM method.

$$\xrightarrow[\text{matrix}]{\text{initial augmented}} \left(\begin{array}{ccc|c} 0.2 & 0.5 & 0.3 & 1710 \\ 0.3 & 0.4 & 0.1 & 1340 \\ 0.1 & 0.6 & 0.4 & 1880 \end{array}\right) \xrightarrow[\text{multiply each row by 10}]{\text{eliminate the decimals}} \left(\begin{array}{ccc|c} 2 & 5 & 3 & 17100 \\ 3 & 4 & 1 & 13400 \\ 1 & 6 & 4 & 18800 \end{array}\right)$$

$$\xrightarrow[\text{with Row 3}]{\text{swap Row 1}} \left(\begin{array}{ccc|c} 1 & 6 & 4 & 18800 \\ 3 & 4 & 1 & 13400 \\ 2 & 5 & 3 & 17100 \end{array}\right) \xrightarrow[\text{to Row 2 add } -2\times\text{Row 1}]{\text{to Row 2 add } -3\times\text{Row 1}} \left(\begin{array}{ccc|c} 1 & 6 & 4 & 18800 \\ 0 & -14 & -11 & -43000 \\ 0 & -7 & -5 & -20500 \end{array}\right)$$

The next step produces decimals. We will round each decimal to four decimal places. For greater accuracy you may use more decimal places.

$$\xrightarrow[\text{by } -14]{\text{divide Row 2}} \left(\begin{array}{ccc|c} 1 & 6 & 4 & 18800 \\ 0 & 1 & 0.7857 & 3071.4286 \\ 0 & -7 & -5 & -20500 \end{array}\right)$$

$$\begin{array}{c} \text{to Row 1 add } -6\times\text{Row 2} \\ \rightarrow \\ \text{to Row 3 add } 7\times\text{Row 2} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -0.7142 & 371.4284 \\ 0 & 1 & 0.7857 & 3071.4286 \\ 0 & 0 & 0.4999 & 1000.0002 \end{array}\right)$$

$$\begin{array}{c} \text{divide Row 3 by 0.4999} \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -0.7142 & 371.4284 \\ 0 & 1 & 0.7857 & 3071.4286 \\ 0 & 0 & 1 & 2000.4005 \end{array}\right)$$

$$\begin{array}{c} \text{to Row 2 add } -0.7857\times\text{Row 3} \\ \rightarrow \\ \text{to Row 1 add } 0.7142\times\text{Row 3} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1800.1144 \\ 0 & 1 & 0 & 1499.7139 \\ 0 & 0 & 1 & 2000.4005 \end{array}\right) \quad \begin{array}{c} \text{solution:} \\ x = 1800.1144 \\ y = 1499.7139 \\ z = 2000.4005 \end{array}$$

Since the final answers are numbers of pieces of furniture, we will round to the nearest whole number. Our solution is to manufacture $x = 1,800$ chairs, $y = 1,500$ cabinets, $z = 2,000$ buffets.

Try it Now Answer

1. $\left(\begin{array}{cc|c} 1 & 1 & 15 \\ 2 & -1 & 12 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 15 \\ 0 & -3 & -18 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 15 \\ 0 & 1 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 6 \end{array}\right)$. Therefore, $x = 9$ and $y = 6$.

Section 3.2 Exercises

In problems 1-2, write an augmented matrix for each system of equations. *Do not solve*.

1. $$\begin{aligned} 2x+5y&=11 \\ x+2y&=8 \end{aligned}$$

2. $$\begin{aligned} 2x+y+3z&=4 \\ 3x-4y+3z&=-6 \\ x+y+z&=3 \end{aligned}$$

In problems 3-4, write the system of equations associated with each augmented matrix.

3. $$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right)$$

4. $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -7 \end{array}\right)$$

In problems 5-15, write an augmented matrix for each system then solve using the Gaussian Elimination Method.

5. $$\begin{aligned} x+y&=5 \\ x-y&=-1 \end{aligned}$$

6. $$\begin{aligned} x+2y&=5 \\ 2x+y&=-2 \end{aligned}$$

7. $$\begin{aligned} 2x-3y&=2 \\ 4x-6y&=7 \end{aligned}$$

8. $$\begin{aligned} x+2y&=1 \\ 2x+4y&=4 \end{aligned}$$

9. $$\begin{aligned} x-y&=1 \\ y-z&=6 \\ x+z&=-1 \end{aligned}$$

10. $$\begin{aligned} 2x+2y-2z&=12 \\ 2x-y+z&=-9 \\ x-2y+3z&=1 \end{aligned}$$

11. $$\begin{aligned} 3x+2y-z&=-16 \\ 6x-4y+3z&=12 \\ 3x+3y+z&=-11 \end{aligned}$$

12. $$\begin{aligned} x+2y+w&=-3 \\ x+2y-z&=6 \\ 2x-y+z+w&=-3 \\ x+2y-w&=3 \end{aligned}$$

13. $$\begin{aligned} 3x-y&=1 \\ 9x-3y&=4 \end{aligned}$$

14. $$\begin{aligned} -x+2y&=5 \\ 4x-8y&=-20 \end{aligned}$$

15.

$$\begin{aligned} 3y+11&=2z \\ 2x&=19+y-3z \\ -8&=3x-2y+z \end{aligned}$$

16. **Manufacturing**: TVPro Inc. produces three models of TV sets: economy, deluxe, and super. Each economy TV set requires 2 hours of electronics work, 2 hours of assembly time, and 1 hour of finishing time. Each deluxe TV set requires 1 hour of electronics work, 3 hours of assembly time, and 1 hour of finishing time. Each super TV set requires 3 hours of electronics work, 2 hours of assembly time, and 2 hours of finishing time. There are 158 hours of labor available for electronics work, 168 hours for assembly, and 104 hours available for finishing each week. How many of each model should be produced each week if all available time for labor is to be used?

Section 3.2 Exercises - Answer Key

1. $\left(\begin{array}{cc|c} 2 & 5 & 11 \\ 1 & 2 & 8 \end{array}\right)$
2. $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 3 & -4 & 3 & -6 \\ 1 & 1 & 1 & 3 \end{array}\right)$
3. $x = 5, y = -2$
4. $x = 4, y = 9, z = -7$
5. $x = 2, y = 3$
6. $x = -3, y = 4$
7. No solution
8. No solution
9. $x = 3, y = 2, z = -4$
10. $x = -1, y = 23, z = 16$
11. $x = -2, y = -3, z = 4$
12. $x = \frac{12}{5} = 2.4, y = -\frac{6}{5} = -1.2, z = -6, w = -3$
13. No solution
14. No solution
15. $x = -4, y = 3, z = 10$
16. 24 economy TV sets, 20 deluxe TV sets, 30 super TV sets

Section 3.3: Matrix Inverse and Matrix Equations

The Multiplicative Identity *I*

The number 1 is called the multiplicative identity for real numbers because any number x multiplied by 1 gives x as an answer (does not change the identity of x): $1 \cdot x = x = x \cdot 1 = x$. Whereas 1 is the multiplicative identity for real numbers, we designate the multiplicative identity for matrices by I. For example, the 2 x 2 identity matrix is as follows.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

I is a square matrix. It has 1s on the main diagonal and 0 entries elsewhere. I is to matrices as 1 is to real numbers. If we multiply a 2 x 2 matrix A by I, then we get the same matrix A for an answer. For example, suppose we multiply the 2 x 2 matrix below

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix}$$

by I. Verify that both $I \cdot A$ and $A \cdot I$ give A for an answer.

$$I \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 0 \cdot (-3) & 1 \cdot 5 + 0 \cdot 4 \\ 0 \cdot 2 + 1 \cdot (-3) & 0 \cdot 5 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = A$$

$$A \cdot I = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 5 \cdot 0 & 2 \cdot 0 + 5 \cdot 1 \\ (-3) \cdot 1 + 4 \cdot 0 & (-3) \cdot 0 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = A$$

Therefore, $I \cdot A = A \cdot I = A$. The 3 x 3 and 4 x 4 identity matrices are shown below.

$$I_{3\text{ x }3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_{4\text{ x }4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Inverse of Matrix

The multiplicative identity 1 plays a key role when solving equations. For example, solve the equation $4x = 12$.

$$\begin{aligned} 4x &= 12 \\ \frac{4x}{4} &= \frac{12}{4} && \text{Divide each side by 4.} \\ 1x &= 3 \\ x &= 3 && \text{Because } 1x = x. \end{aligned}$$

Now suppose A, X, and B are matrices. How would we solve the matrix equation

$$AX = B$$

for the matrix X? We cannot divide each side by the matrix A because *there is no division of matrices*!

To see what we do, let's return to basic algebra. How would we solve the equation *without division*? We would use multiplication. Instead of dividing each side by 4, we multiply each side by the **multiplication inverse** of 4, which is ¼. Look again at the solution of $4x = 12$.

$$\begin{aligned} 4x &= 12 \\ \frac{1}{4} \cdot 4x &= \frac{1}{4} \cdot 12 && \text{Multiply each side by the multiplicative inverse of 4 or 1/4.} \\ 1x &= 3 && \text{Since } 1/4 \cdot 4 = 1. \\ x &= 3 && \text{Because } 1x = x. \end{aligned}$$

The product of a number and its multiplicative inverse is 1. Verify on your graphing calculator that $1/4 \cdot 4 = 1$. Also verify that $1/4 \cdot 12 = 3$.

Notation for the Inverse of a Matrix

Above we introduced the notation x^{-1} because it is used when speaking about the inverse of a matrix. Just as 4^{-1} is the notation we use for the multiplicative inverse of 4, the multiplicative inverse of the matrix A is written A^{-1}. We read A^{-1} as "A inverse." A^{-1} does not mean 1/A. It simply denotes the inverse of A.

And just as $4^{-1} \cdot 4 = 1$, the product of the matrix A and its inverse A^{-1} is the identity matrix I.

$$A^{-1} \cdot A = I \qquad \text{and} \qquad A \cdot A^{-1} = I$$

Before we can verify these equations, we need a procedure for finding the inverse of a matrix.

Finding the Inverse of a Matrix A

Only square matrices have inverses. We will use the Gauss-Jordan Method to find the inverse A^{-1} of the matrix A, if it exists. We illustrate the process in the next two examples.

Example 1

Use the Gauss-Jordan Method to find the inverse of the 2 x 2 matrix A.

$$A=\begin{pmatrix}5&-2\\3&4\end{pmatrix}$$

Solution: We begin by forming the augmented matrix $(A|I)$ where A is on the left side of the vertical bar and the 2 x 2 identity matrix I is in the right side. Apply the three row operations to the left side of the bar until the identity matrix I is on the left side. We then get $(I\,|\,A^{-1})$. The inverse of A, if it exists, will be on the right side of the bar.

$$\left(\begin{array}{cc|cc}5&-2&1&0\\3&4&0&1\end{array}\right)\xrightarrow[\text{to transform into}]{\text{apply the three row operations}}\left(\begin{array}{cc|c}1&0&\text{inverse}\\0&1&\text{matrix}\end{array}\right)$$

$$A \qquad I \qquad\qquad\qquad\qquad\qquad\qquad I \qquad A^{-1}$$

The work is shown below.

$$\left(\begin{array}{cc|cc}5&-2&1&0\\3&4&0&1\end{array}\right)\xrightarrow[\text{to Row 1 add }-2\times\text{Row 2}]{\text{change 5 into a 1}}\left(\begin{array}{cc|cc}-1&-10&1&-2\\3&4&0&1\end{array}\right)$$

$$\xrightarrow[\text{Row 1 by }-1]{\text{multiply}}\left(\begin{array}{cc|cc}1&10&-1&2\\3&4&0&1\end{array}\right)\xrightarrow[\text{to Row 2 add }-3\times\text{Row 1}]{\text{change 3 into a 0}}\left(\begin{array}{cc|cc}1&10&-1&2\\0&-26&3&-5\end{array}\right)$$

$$\xrightarrow[\text{divide Row 2 by }-26]{\text{change }-26\text{ into 1}}\left(\begin{array}{cc|cc}1&10&-1&2\\0&1&-\frac{3}{26}&\frac{5}{26}\end{array}\right)\xrightarrow[\text{to Row 1 add }-10\times\text{Row 2}]{\text{change 10 into a 0}}\left(\begin{array}{cc|cc}1&0&\frac{2}{13}&\frac{1}{13}\\0&1&-\frac{3}{26}&\frac{5}{26}\end{array}\right)$$

$$I \qquad A^{-1}$$

Use your calculator to help you perform operations involving fractions. For example, in the last row above, to change -1 in the left matrix into 2/13 in the right matrix, enter $-10\bullet(-3/26)+(-1)$ into your calculator as follows:

(-) 10 × (-) 3 ÷ 26 + (-) 1

Please check your user manual to find out how to convert this to a fraction!

From above, the inverse of the matrix $A=\begin{pmatrix}5 & -2\\ 3 & 4\end{pmatrix}$ is the matrix $A^{-1}=\begin{pmatrix}\frac{2}{13} & \frac{1}{13}\\ -\frac{3}{26} & \frac{5}{26}\end{pmatrix}$. To check, multiply $A\cdot A^{-1}$ and show that the answer is the 2 x 2 identity matrix I.

$$A\cdot A^{-1}=\begin{pmatrix}5 & -2\\ 3 & 4\end{pmatrix}\cdot\begin{pmatrix}\frac{2}{13} & \frac{1}{13}\\ -\frac{3}{26} & \frac{5}{26}\end{pmatrix}=\begin{pmatrix}5\cdot\frac{2}{13}+(-2)\cdot-\frac{3}{26} & 5\cdot\frac{1}{13}+(-2)\cdot\frac{5}{26}\\ 3\cdot\frac{2}{13}+4\cdot-\frac{3}{26} & 3\cdot\frac{1}{13}+4\cdot\frac{5}{26}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

You can verify that $A^{-1}\cdot A=I$.

Example 2

Find the inverse A^{-1} of the matrix $A=\begin{pmatrix}2 & 0 & 4\\ 3 & 1 & 5\\ -1 & 1 & -2\end{pmatrix}$

Solution: Begin by writing the augmented matrix $(A|I)$. The matrix A is on the left side of the vertical bar and the 3 x 3 identity matrix I is on the right side.

$$\underbrace{\left(\begin{array}{ccc|ccc}2 & 0 & 4 & 1 & 0 & 0\\ 3 & 1 & 5 & 0 & 1 & 0\\ -1 & 1 & -2 & 0 & 0 & 1\end{array}\right)}_{}\qquad \begin{array}{cc}A & \qquad I\end{array}$$

Apply the three row operations to reduce this matrix to the form $(I\,|\,A^{-1})$, where the 3 x 3 identity matrix I is on the left side of the vertical bar. Then the inverse of A, if it exists, will be on the right side.

$$\xrightarrow{\text{Row } 1 \div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 3 & 1 & 5 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{to Row 3 add } 1\times \text{Row } 1]{\text{to Row 2 add } -3\times \text{Row } 1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{swap Row 2 with Row 3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & -1 & -\frac{3}{2} & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\text{to Row 3 add } -1\times \text{Row } 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{-1\times \text{Row } 3} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\text{to Row 1 add } -2\times \text{Row } 3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{2} & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$I \qquad\qquad A^{-1}$$

The left side of the vertical bar is the 3 x 3 identity matrix I and the right side is the inverse of the A matrix, which is

$$A^{-1} = \begin{pmatrix} -7/2 & 2 & -2 \\ 1/2 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

You can verify that this is the inverse of A by showing that both $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are equal to I.

Example 3

Find the inverse A^{-1} of the matrix: $A = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$.

Solution: Write the augmented matrix $(A \mid I)$ and use the three row operations to reduce it to the form $(I \mid A^{-1})$.

$$(A \mid I) = \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{to Row 2 add } 3\times\text{Row 1}} \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

Because each entry in the bottom row to the left of the vertical bar is 0, we stop. The 0s tell us that the matrix A has no inverse. It is not possible to install the 2 x 2 identity matrix on the left side of the vertical bar. The answer is “A has no inverse.” A matrix that does not have an inverse is called a **singular** matrix.

Try it Now 1

Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$.

Solving the Matrix Equation $A \cdot X = B$

Suppose we are given the matrix equation $A \cdot X = B$, where X is a matrix of variables. We will solve for X by first finding the inverse A^{-1} of the matrix A. Then we multiply each side of the equation by A^{-1}, *on the left side*. Why on the left? Recall, what we do to one side of an equation we must also do to the other side. Because matrix multiplication is not commutative, if we multiply on the left on one side, we must also multiply on the left on the other side.

$A \cdot X = B$

$A^{-1} \cdot A \cdot X = A^{-1} \cdot B$ Multiply each side of the equation by A^{-1}, the inverse of A, on the left side.

$I \cdot X = A^{-1} \cdot B$ Because A^{-1} and A are inverses, $A^{-1} \cdot A = I$.

$X = A^{-1} \cdot B$ The matrix I is the multiplicative identity, so $I \cdot X = X$. This is the solution.

Therefore, **the solution of the matrix equation** $A \cdot X = B$ is $X = A^{-1} \cdot B$.

Or simply put the product $A^{-1} \cdot B$ is the solution of the equation $A \cdot X = B$. We will use matrix inverses and this procedure to solve systems of equations.

Solving Systems of Equations Using the Matrix Inverse

Example 4

Use matrices to solve the system of two equations in two unknowns shown here.

$$\begin{aligned} 5x - 2y &= 12 \\ 3x + 4y &= 2 \end{aligned} \qquad (1)$$

Solution: Use the coefficients to make the matrix $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$. Use the variables to make the matrix $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Use the constants to make the matrix $B = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$. We may then write the system of equations in (1) as a matrix equation in the form $A \cdot X = B$ as follows.

$$A \quad \cdot \quad X \quad = \quad B$$

$$\underbrace{\overset{\begin{matrix} x & \;\; y \end{matrix}}{\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}}}_{\substack{\text{Coefficient} \\ \text{Matrix A}}} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\substack{\text{Variable} \\ \text{Matrix X}}} = \underbrace{\begin{pmatrix} 12 \\ 2 \end{pmatrix}}_{\substack{\text{Constants} \\ \text{Matrix B}}} \qquad (2)$$

The coefficient matrix A contains like coefficients in the same column. The x-coefficients are in the first column and the y-coefficients are in the second column. The top to bottom order of the variables in matrix X must be in the same order as the column labels in matrix A.

Note, we may transform the matrix equations (2) back to the system of equations (1) by multiplying $A \cdot X$ on the left side of the equation (2). We get $\begin{pmatrix} 5x - 2y \\ 3x + 4y \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$. Each side of the equation is a 2 x 1 matrix. Since they are equal corresponding entries are equal.

Therefore, the 1st row 1st column of each matrix gives us the first equation $5x-2y=12$ in (1) and the 2nd row 1st column in each gives us the second equation $3x+4y=2$.

To solve the matrix equation in (2), first find the inverse of matrix A. We already did this in Example 1. The inverse is $A^{-1}=\begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix}$. To solve the matrix equation, multiply each side of (2) by this inverse. The entire solution is below.

$$\begin{array}{ccccc} A & \cdot & X & = & B \end{array}$$

$$\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix} \tag{2}$$

Multiply each side by the inverse A^{-1} on the left side.

$$\begin{array}{ccccccccc} A^{-1} & \cdot & A & \cdot & X & = & A^{-1} & \cdot & B \end{array}$$

$$\begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

Verify for yourself that $A^{-1} \cdot A = I$ on the left side, so that we get

$$\begin{array}{ccccccc} I & \cdot & X & = & A^{-1} & \cdot & B \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

Verify on the left side that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and verify on the right side that $\begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Finally we get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Since corresponding entries are equal, our final answer is $x=2$ and $y=-1$.

Try it Now 2

Use matrices to solve the system of equations: $\begin{aligned}5x+3y&=18\\3x+2y&=11\end{aligned}$.

Solving Larger Systems of Equations

We will use the procedure illustrated in the previous examples to solve larger systems of equations.

Example 5

Use matrices to solve the following system: $\begin{aligned}5x+2y-3z&=10\\2x-z&=-4\\-x+3y+2z&=5\end{aligned}$.

Solution:

First, build the coefficient matrix A, the variable matrix X, and the constants matrix B.

$$A=\begin{array}{c}\begin{array}{ccc}x & y & z\end{array}\\ \begin{pmatrix}5 & 2 & -3\\2 & 0 & -1\\-1 & 3 & 2\end{pmatrix}\end{array} \qquad X=\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad B=\begin{pmatrix}10\\-4\\5\end{pmatrix}$$

Second, write the matrix equation for this system in the form $A\cdot X=B$.

$$\underset{A}{\begin{pmatrix}5 & 2 & -3\\2 & 0 & -1\\-1 & 3 & 2\end{pmatrix}}\cdot\underset{X}{\begin{pmatrix}x\\y\\z\end{pmatrix}}=\underset{B}{\begin{pmatrix}10\\-4\\5\end{pmatrix}}$$

Third, find the inverse of A using either the Gauss-Jordan Method, or your scientific calculator. We get

$$A^{-1}=\begin{pmatrix}-\frac{1}{3} & \frac{13}{9} & \frac{2}{9}\\ \frac{1}{3} & -\frac{7}{9} & \frac{1}{9}\\ -\frac{2}{3} & \frac{17}{9} & \frac{4}{9}\end{pmatrix}$$

Fourth, calculate the solution of the system, which is given by $X = A^{-1} \cdot B$.

$$\begin{array}{ccccc} X & = & A^{-1} & \cdot & B \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{13}{9} & \frac{2}{9} \\ \frac{1}{3} & -\frac{7}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{17}{9} & \frac{4}{9} \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \\ -12 \end{pmatrix}$$

$$\text{Or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \\ -12 \end{pmatrix}$$

Therefore, $x = -8$, $y = 7$, and $z = -12$.

You can check your answer by substituting these values into each of the original three equations (and can use the calculator to speed up calculations).

To use matrices to solve a system of equations, we must first write each equation in standard form, similar to the approach in the previous section. That is, write the variables on the left side of the equals sign and the constants on the right side.

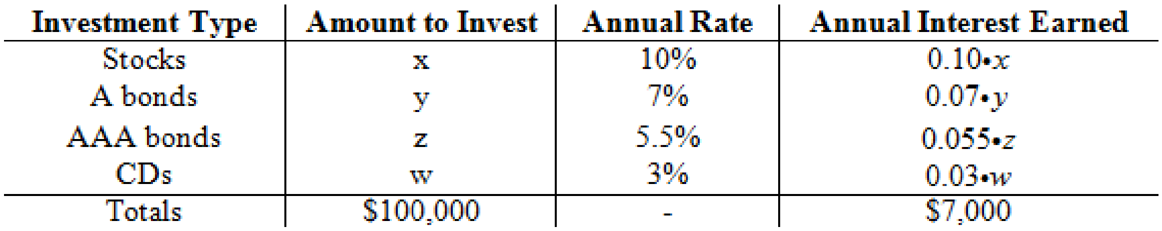
Example 6

Leslie plans to retire in ten years. She has $100,000 in savings. She plans to invest this money in relatively safe areas. She wants to earn interest income of $7,000 each year. To diversify, she plans to invest in four areas:

Stocks earning 10% annually
A rated bonds earning 7% annually
AAA rated bonds earning 5.5% annually
CDs earing 3% annually

The amount she invests in stocks will equal the total invested in AAA bonds and CDs. Also, she wants to invest $5,000 more in AAA bonds than in A bonds. How much should Leslie invest in each of the four areas?

Solution: *First, build a table* summarizing all of the given information. Since we want to invest a portion of the $100,000 in four different areas, we will need four variables.



Investment Type	Amount to Invest	Annual Rate	Annual Interest Earned
Stocks	x	10%	$0.10\cdot x$
A bonds	y	7%	$0.07\cdot y$
AAA bonds	z	5.5%	$0.055\cdot z$
CDs	w	3%	$0.03\cdot w$
Totals	$100,000	-	$7,000

amount in stocks = amount in AAA bounds + amount in CDs
invest $5,000 more in AAA bonds than in A bonds

Note, *x* dollars of the $100,000 total is invested in stocks. Since stocks earn 10% per year, the annual interest earned from stocks of 10% of *x* or $0.10\cdot x$ dollars.

Second, build equations using the given information. Two columns in the table give us equations. The “Amount to Invest” column gives us one equation. Since the amounts invested in each of the four areas must add up to $100,000, we have

$$x+y+z+w=100,000$$

The “Annual Interest Earned” column gives us a second equation. Since the sum of the interests earned in each of the four investments must total $7,000, we have

$$0.10x+0.07y+0.055z+0.03w=7,000$$

The next two equations come from conditions Leslie imposed on her investments. She said the following.

$$\begin{array}{ccccc} \text{amount in stocks} & = & \text{amount in AAA bounds} & + & \text{amount in CDs} \\ x & = & z & + & w \end{array}$$

or $x=z+w$. To use this equation in a matrix we must first write it in standard form (variables on the left side and constants on the right side). To do this subtract *z* from each side and subtract *w* from each side. We get

$$x-z-w=0$$

The last equation comes from Leslie’s final condition. She said the following.

invest $5,000 more in AAA bonds than in A bonds

In other words, z (amount to invest in AAA bonds) is $5,000 more than y (amount to invest in A bonds). So if we subtract $5,000 from z we get y.

$$z-5,000=y$$

To put this equation in standard form, add $5,000 to each side and subtract y from each side. We get

$$-y+z=5,000$$

In summary, we have the following system of four equations in four unknowns.

$$\begin{aligned} x+y+z+w&=100,000 \\ 0.10x+0.07y+0.055z+0.03w&=7,000 \\ x-z-w&=0 \\ -y+z&=5,000 \end{aligned}$$

We will solve this system using matrix equations. *Third, write a matrix equation in the form* $A \cdot X = B$. Recall A is the matrix of coefficients. The first column is x coefficients, the second column is y coefficients, etc. The matrix of variables is X and the matrix of constants is B.

$$\begin{array}{ccccc} A & \cdot & X & = & B \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.10 & 0l07 & 0.055 & 0.03 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} & = & \begin{pmatrix} 100000 \\ 7000 \\ 0 \\ 5000 \end{pmatrix} \end{array}$$

Find the inverse of A matrix using either the Gauss-Jordan Method or the calculator. We get

$$A^{-1}=\begin{pmatrix} \frac{19}{12} & -\frac{50}{3} & \frac{13}{12} & \frac{5}{12} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & -\frac{5}{6} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & \frac{1}{6} \\ \frac{15}{4} & -50 & \frac{5}{4} & \frac{1}{4} \end{pmatrix}$$

The solution of the system is $X = A^{-1} \cdot B$.

$$\begin{array}{ccccccc} X & = & A^{-1} & \cdot & B & & \\ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} & = & \begin{pmatrix} \frac{19}{12} & -\frac{50}{3} & \frac{13}{12} & \frac{5}{12} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & -\frac{5}{6} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & \frac{1}{6} \\ \frac{15}{4} & -50 & \frac{5}{4} & \frac{1}{4} \end{pmatrix} & \cdot & \begin{pmatrix} 100000 \\ 7000 \\ 0 \\ 5000 \end{pmatrix} & = & \begin{pmatrix} 43750 \\ 12500 \\ 17500 \\ 26250 \end{pmatrix} \end{array}$$

The solution is $x = 43750$, $y = 12500$, $z = 17500$, and $w = 26250$.

Leslie should invest $43,750 in stocks, $12,500 in A rated bonds, $17,500 in AAA rated bonds, and $26,250 in CDs in order to make $7,000 in interest earnings annually.

To check, return to the original word problem and make sure the answers satisfy all the conditions. For example, the sum of the four amounts $43,750 + $12,500 + $17,500 + $25,250 = $100,000, the total she had to invest. Take time and check the other conditions yourself.

If the answer $X = A^{-1} \cdot B$ in Example 6 contained a negative number, then there would be no practical solution to Leslie's investment plan. We cannot invest a negative amount. Leslie would have to change at least one of her conditions and start over again.

Try it Now Answers

1.

$$(I|A) = \left(\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -7 & 3 \end{array}\right) = \left(\begin{array}{cc|cc} \frac{1}{5} & 0 & 1 & -\frac{2}{5} \\ 0 & 1 & -7 & 3 \end{array}\right) = \left(\begin{array}{cc|cc} \frac{1}{5} & 0 & 1 & -\frac{2}{5} \\ \frac{7}{5} & 1 & 0 & \frac{1}{5} \end{array}\right) = \left(\begin{array}{cc|cc} \frac{1}{5} & 0 & 1 & -\frac{2}{5} \\ 7 & 5 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 7 & 5 & 0 & 1 \end{array}\right) = (A^{-1}|I)$$

So, therefore the inverse matrix is $A^{-1} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$.

2. Convert the system $\begin{matrix} 5x + 3y = 18 \\ 3x + 2y = 11 \end{matrix}$ into matrices.

$$\begin{array}{ccccc} A & \cdot & X & = & B \\ \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 18 \\ 11 \end{pmatrix} \end{array}$$

Now find the inverse of matrix A.

$$A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

Calculate the solution of the system which is given by $X = A^{-1} \cdot B$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 11 \end{pmatrix} = \begin{pmatrix} (2 \cdot 18) + (-3 \cdot 11) \\ (-3 \cdot 18) + (5 \cdot 11) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Therefore, $x = 3$ and $y = 1$.

Section 3.3 Exercises

In problems, 1-4, use matrix multiplication to decide if the matrices are inverses of each other (check to see if their product is the identity matrix I).

1. $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

In problems 5-10, find the inverse of each matrix, if it exists. Express all entries in fraction form.

5.
$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

6.
$\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

9. $\begin{pmatrix} 1 & -2 & 3 & 0 \\ -2 & 2 & -2 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

10. $\begin{pmatrix} -2 & -1 & 3 & 1 \\ 1 & 2 & -2 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$

In problems 11-21, solve each system of equations by finding the inverse of the coefficient matrix.

11.
$$\begin{aligned} 2x+3y&=10 \\ x-y&=-5 \end{aligned}$$

12.
$$\begin{aligned} 3x-5y&=-1 \\ 2x+3y&=12 \end{aligned}$$

13.
$$\begin{aligned} 4x+5y&=9 \\ 3x-2y&=-2 \end{aligned}$$

14.
$$\begin{aligned} 5x-3y&=14 \\ 3x+4y&=6 \end{aligned}$$

15.
$$\begin{aligned} 2x+y&=3 \\ -4x+2y&=1 \end{aligned}$$

16.
$$\begin{aligned} -x-4y&=1 \\ 3x-12y&=5 \end{aligned}$$

17.
$$\begin{aligned} 4x+5y&=-2 \\ x+y+z&=-1 \\ y-3z&=3 \end{aligned}$$

18.
$$\begin{aligned} x+2y+3z&=2 \\ x+4y+3z&=-8 \\ y-z&=-4 \end{aligned}$$

19.
$$\begin{aligned} 2x-3y+4z&=5 \\ 3x+y-2z&=-3 \\ 4x+2y+3z&=1 \end{aligned}$$

20. **Bond Investment**: An investment company recommends that a client invest in AAA, AA, and A rated bonds. The average annual yield on AAA bonds is 6%, on AA bonds 7%, and on A bonds 10%. The client tells the company she wants to invest twice as much in AAA bonds as in A bonds. How much should be invested in each type of bond under the following conditions?

a) The total investment is \$50,000 and the investor wants an annual income (that is, earned interest) of \$3, 620.
b) The total investment is \$15,000 yielding an annual income of \$1,075.
c) The total investment is \$21,475 yielding \$1,630 annual income. Is this workable? Explain. What happens if we replace \$1,630 by \$1,550?

Section 3.3 Exercises – Answer Key

1. Yes
2. Yes
3. No
4. No
5. $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$
6. $\begin{pmatrix} -5 & 2 \\ -3 & 1 \end{pmatrix}$
7. $\begin{pmatrix} \frac{2}{11} & -\frac{1}{22} & -\frac{7}{11} \\ \frac{2}{11} & -\frac{1}{22} & \frac{4}{11} \\ \frac{1}{11} & \frac{5}{22} & \frac{2}{11} \end{pmatrix}$
8. $\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{9}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$
9. $\begin{pmatrix} 3 & 1 & 3 & -2 \\ 4 & 2 & 3 & -1 \\ 2 & 1 & 1 & 0 \\ -2 & -1 & -2 & 2 \end{pmatrix}$
10. $\begin{pmatrix} -\frac{4}{5} & -\frac{6}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{3}{10} & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{5} & -\frac{7}{10} & -\frac{3}{10} & \frac{11}{10} \end{pmatrix}$
11. $x = -1, y = 4$
12. $x = 3, y = 2$
13. $x = \frac{8}{23}, y = \frac{35}{23}$
14. $x = \frac{74}{29}, y = -\frac{12}{29}$

15. $x = \frac{5}{8}, y = \frac{7}{4}$
16. $x = \frac{1}{3}, y = -\frac{1}{3}$
17. $x = -8, y = 6, z = 1$
18. $x = 15, y = -5, z = -1$
19. $x = -\frac{14}{73}, y = -\frac{39}{73}, z = \frac{69}{73}$
20.
    a. $24,000 in AAA bonds, $14,000 in AA bonds, $12,000 in A bonds.
    b. $5,000 in AAA bonds, $7,500 in AA bonds, $2,500 in A bonds.
    c. No, it is not possible to have a total investment of $21,475 yielding $1,630 annual income because the amounts would have to be $25,350 in AAA bonds, -$16,550 in AA bonds, and $12,675 in A bonds. It is not possible to invest a negative amount of money. It is possible if the annual income is $1,550. The amounts would have to be $9,350 in AAA bonds, $7,450 in AA bonds, and $4,675 in A bonds.

Chapter 4: Linear Programming

Section 4.1: Graphing Linear Equations and Inequalities

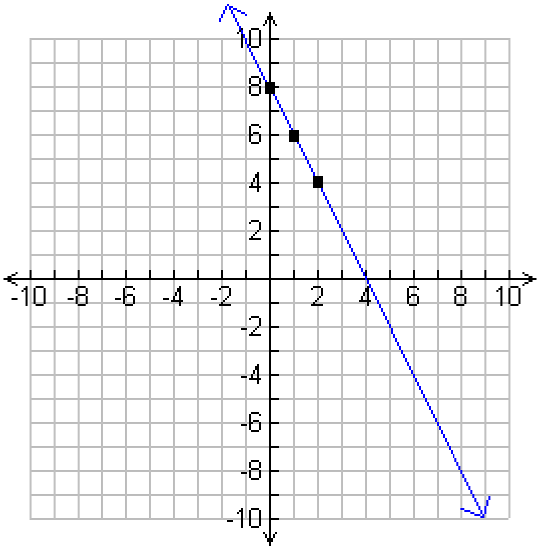
A **linear equation in two variables** is one that can be written in the form $ax+by=c$ where a, b, and c are real numbers but $a \neq 0$ and $b \neq 0$. There are several methods that can be used to graph linear equations in two variables. Two methods will be reviewed: plotting points and graphing by intercepts.

Graphing by Plotting Points

The **plotting points method** can be used in any case. You choose random numbers for one variable and then solve for the other. Each time this is complete, you have found a point along the graph of the line. You repeat this at least once to get a second point. Finding a third point serves as a check. The example below uses three points. If the three points do not form a straight line, return and check your work.

Example 1

Graph $2x+y=8$ by plotting points.

<table>
<tr><th>Procedure
(in words)</th><th>Procedure
(in action)</th><th>Graph</th></tr>
<tr>
<td>Pick any random value for x or y. Substitute this value into the equation and solve for the remaining variable. You now have one point that is a part of the graph. Repeat for more points (at least two total).</td>
<td>
<table>
<tr><th>x</th><th>y</th><th>(x, y)</th></tr>
<tr><td>0</td><td>$2(0)+y=8$
$0+y=8$
$y=8$</td><td>$(0,8)$</td></tr>
<tr><td>1</td><td>$2(1)+y=8$
$2+y=8$
$y=6$</td><td>$(1,6)$</td></tr>
<tr><td>2</td><td>$2(2)+y=8$
$4+y=8$
$y=4$</td><td>$(2,4)$</td></tr>
</table>
</td>
<td> </td>
</tr>
</table>

The plotting points method can be used at any given time, but it is sometimes more tedious than other methods. This method isn't particularly useful when graphing curves, although those graphs are not the focus of this course.

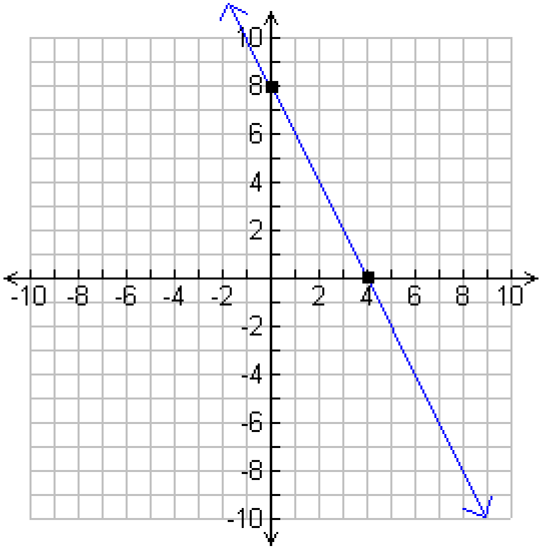
Try it Now 1
Graph $y = 5x - 3$ by plotting points.

Graphing Using Intercepts

The next method discussed is the **intercept method**. Recall that the *x*-intercept of the graph of an equation is the point(s) where the graph intersects the *x*-axis. Similarly, the *y*-intercept of the graph of an equation is the point(s) where the graph intersects the *y*-axis. This method is mostly convenient for graphing lines, but for graphs of other equations that we won't examine in this course (such as circles), these intercepts sometimes don't exist. Let's review graphing the equation of a line using the intercept method.

Example 2

Graph $2x + y = 8$ using the intercept method.

Procedure (in words)	**Procedure (in action)**	**Graph**
To find the *x*-intercept, the point where the graph intercepts the *x* axis, let $y = 0$. Solve for *x*. You now have a coordinate pair representing the *x*-intercept.	x-intercept $2x + 0 = 8$ $2x = 8$ $x = 4$ $(4, 0)$	
To find the *y*-intercept, the point where the graph intercepts the *y* axis, let $x = 0$. Solve for *y*. You now have a coordinate pair representing the *y*-intercept.	y-intercept $2(0) + y = 8$ $0 + y = 8$ $y = 8$ $(0, 8)$	

Notice the equation used on Example 1 and Example 2 were the same. We used two different graphing techniques and both resulted in the same graph. The method you decide to use, in upcoming sections, will be up to you.

Try it Now 2
Graph $-3x + 12y = 12$ using the intercept method.

An **inequality** is a statement containing one of these symbols: $<, \leq, >, \geq$. When we graph a linear inequality, we have an infinite solution set. For example, consider the inequality $x + y > 9$. One possible solution is $(10, 12)$ since $10 + 12 > 9$. Another solution is $(-8, 20)$ since $-8 + 20 > 9$. We could list ordered pairs that are solutions to this inequality, for the rest of our lives! After all,

$(1000000, 300000000)$ is a solution too! When we graph an inequality, we are constructing a visual representation of all possible solutions. We will examine a few examples.

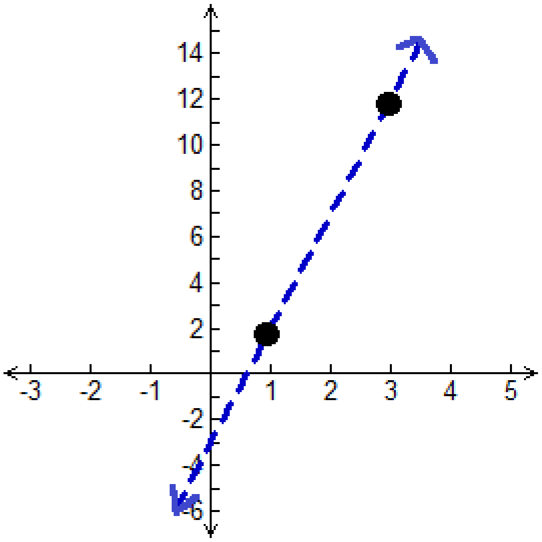
Example 3

Graph $y > 5x - 3$.

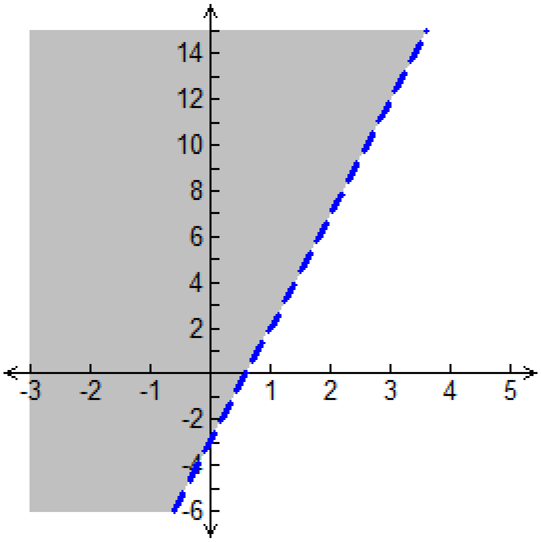
We first graph the equation $y = 5x - 3$. Let's use the plotting points method, where we choose (at least) two random numbers, in the case for *x*, and solve for *y*.

x	y	(x, y)
1	$y = 5(1) - 3 = 5 - 3 = 2$	$(1, 2)$
3	$y = 5(3) - 3 = 15 - 3 = 12$	$(3, 12)$

We will plot these two points on a graph and then begin sketching the line. Typically, we draw a solid line. However, inequalities are different. Points along this line don't actually satisfy the original inequality. For example, consider the point $(3,12)$. When you substitute these values into the original inequality, you get $12 > 5(3) - 3$ or $12 > 12$. This is false. Therefore, we will use a dotted line when connecting the points. Whenever you have an inequality with a < or > symbol, a dashed line will always be used. Solid lines with be used with ≤ and ≥ symbols and we'll examine this in the next example.



We're not quite finished with the graph. We have to determine which side of the line represents the solution set. To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0, 0)$ because it is easy to work with: $0 > 5(0) - 3$. This results in $0 > -3$, which is a true statement. This means that every point on the same side of the line as $(0, 0)$ satisfies the inequality. (Try a few if you don't believe it. Every point on the other side of the line makes the inequality false.) We want to shade the side of the line that represents the solution set. Therefore, we shade the northwestern part of the graph.



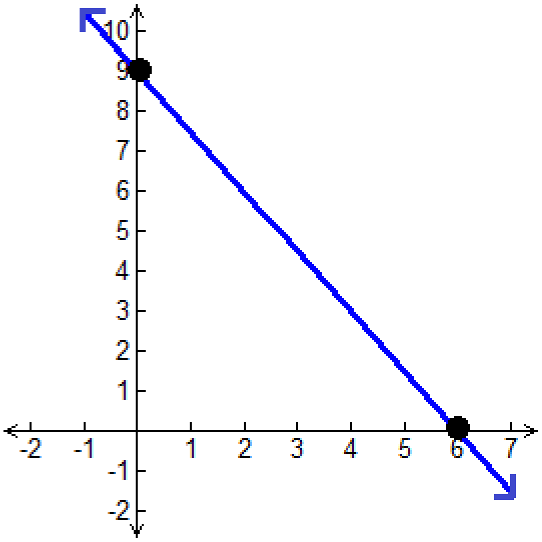
Example 4

Graph $3x+2y \geq 18$ using the intercept method.

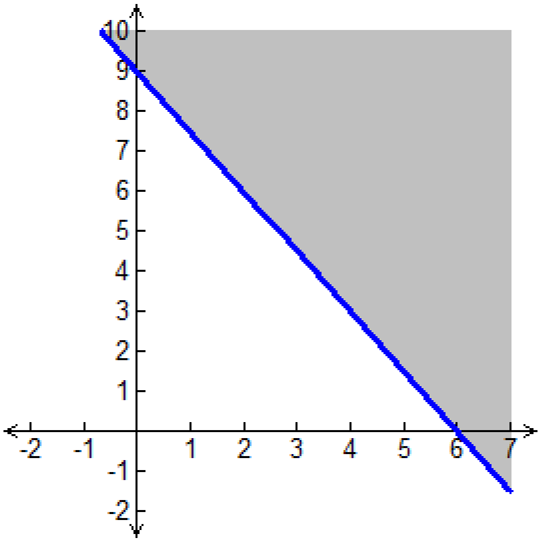
We first graph the equation $3x+2y=18$. For variety, let's graph this using the intercept method (although the plotting points method will work too).

x	**y**	(x, y)
0	$3(0)+2y=18$ $2y=18$ $y=9$	$(0,9)$
$3x+2(0)=18$ $3x=18$ $x=6$	0	$(6,0)$

Point the points on a graph. In the previous example, we connected these with a dashed line. This inequality contains a ≤ symbol, meaning the points we determined (the intercepts) actually satisfy the original inequality. Try one: $3(0)+2(9) \geq 18$ gives $18 \geq 18$, which is a true statement. Therefore, a solid line will be used. Remember, whenever you have an inequality with a < or > symbol, a dashed line will always be used and whenever you have an inequality with a ≤ or ≥ symbol, a solid line will be used.



Now we are finished graphing the equation (the line). Return to the inequality $3x+2y\geq 18$. To determine which side of the line, test any point not on the line by substituting its coordinates for x and y in the inequality. Suppose we test the point $(0,0)$ because it is easy to work with: $3(0)+2(0)\geq 18$. This results in $0\geq 18$, which is a false statement. This means that every point on the same side of the line as $(0,0)$ *does not* satisfy the inequality. (Try a few if you don't believe it. Every point on the other side of the line makes the inequality true.) We want to shade the side of the line that does represent the solution set. Therefore, we shade the northeastern part of the graph.



Try it Now 3

Graph $4x+12y\leq 24$.

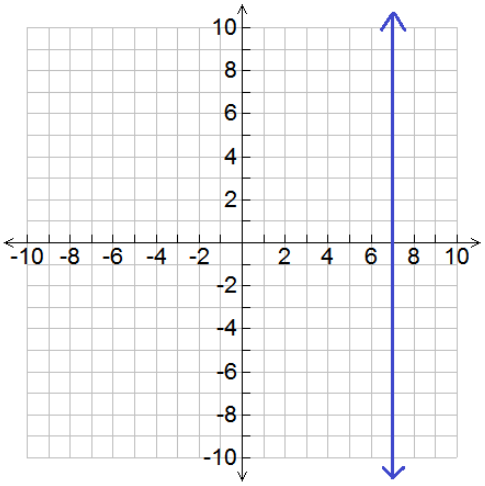
Graphing Horizontal and Vertical Lines

At the beginning of this section, we defined a linear equation in two variables is one that can be written in the form $ax+by=c$ where *a*, *b*, and *c* are real numbers but $a \neq 0$ and $b \neq 0$. If *a* or *b* is zero, we have a linear equation in one variable. Graphs of these equations are still linear but provide horizontal or vertical lines.

Example 5

Graph $x=7$.

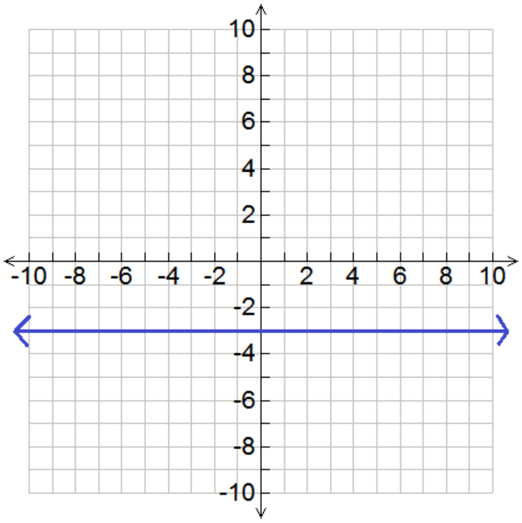
The graph of this equation represents the set of points where $x=7$. Ordered pairs where $x=7$ include $(7,-4)$, $(7,0)$, or $(7,6)$. The *y* values can include any real number. If you plot these points, you will notice a vertical line graph forming:



Example 6

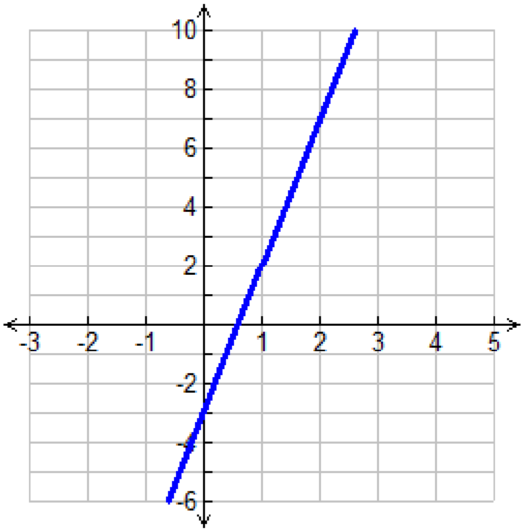
Graph $y=-3$.

The graph of this equation represents the set of points where $y=-3$. Ordered pairs where $y=-3$ include $(-4,-3)$, $(0,-3)$, or $(5,-3)$. The *x* values can include any real number. If you plot these points, you will notice a horizontal line graph forming:

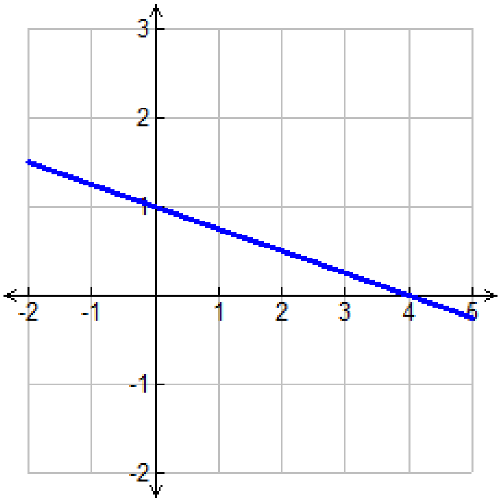


Try it Now Answers

1.



2.



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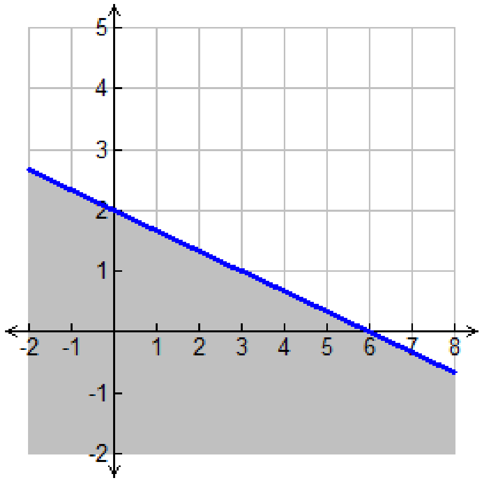


Section 4.1 Exercises

Graph the linear equation by plotting points.

1. $y = 3x - 7$
2. $y = -9x + 12$
3. $3x + y = 10$
4. $y = 5x + 9$

Graph the linear equation using the intercept method.

5. $4x - 2y = 12$
6. $5x - 10y = 20$
7. $x + y = -4$
8. $y = 3x - 15$

Graph the linear inequality.

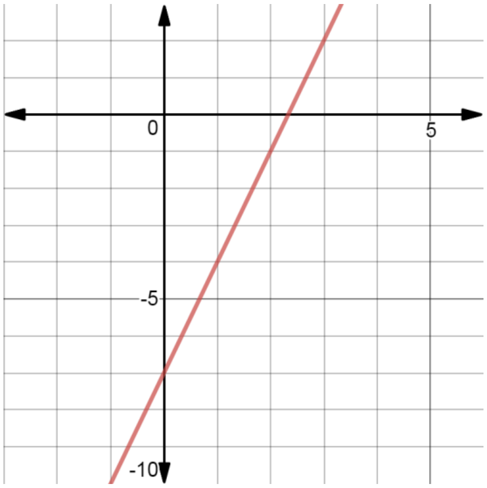
9. $y > 3x - 9$
10. $y \le x + 7$
11. $3y - x \le -6$
12. $10x - 30y \ge 60$

Graph the horizontal or vertical line.

13. $x = -9$
14. $y = 1$

Section 4.1 Exercises – Answer Key

1.



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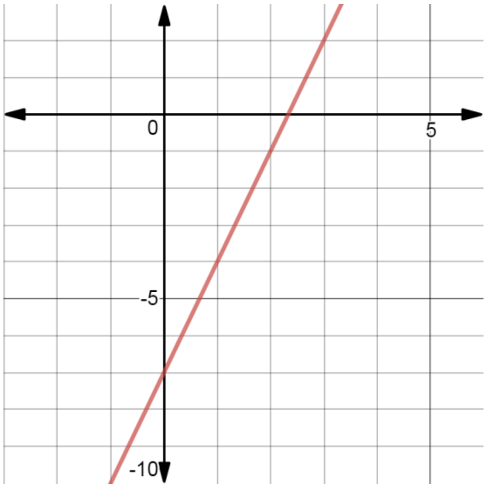

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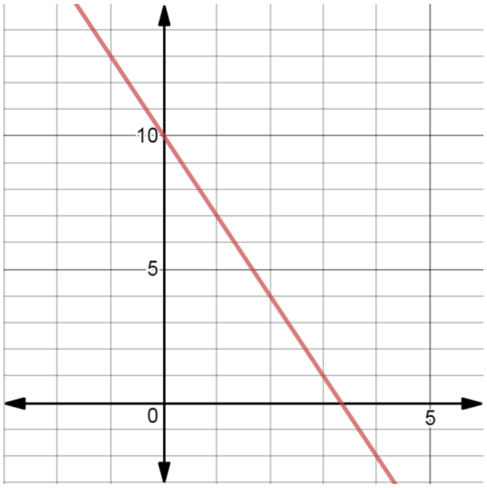

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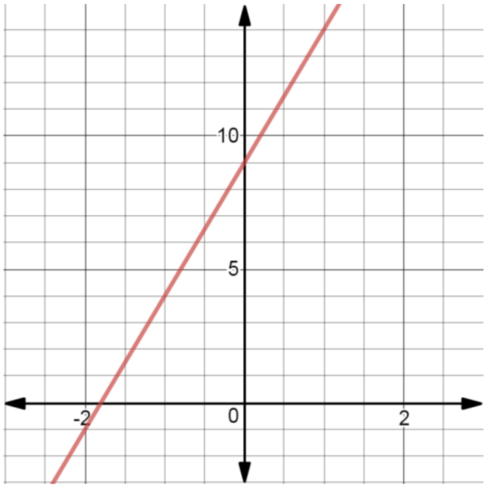

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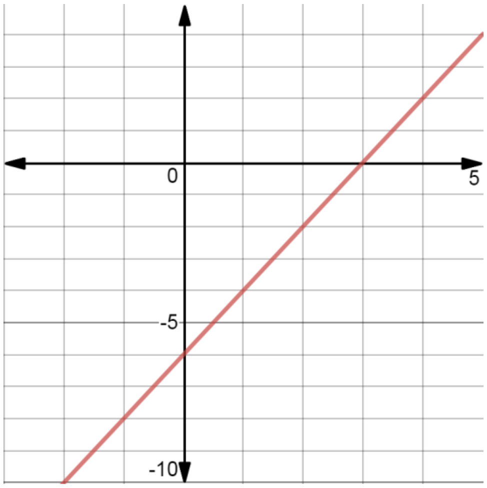

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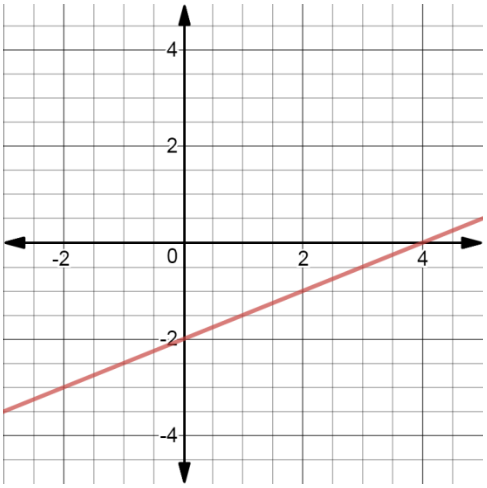

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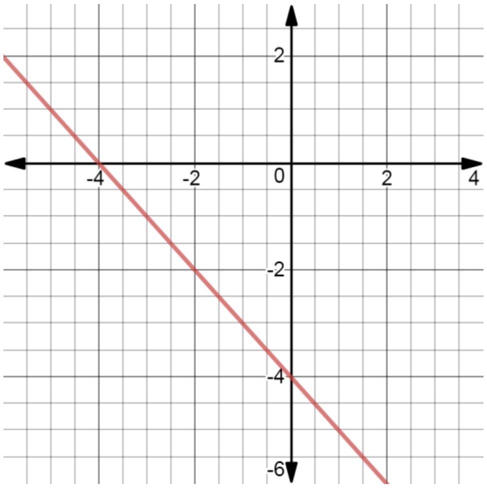

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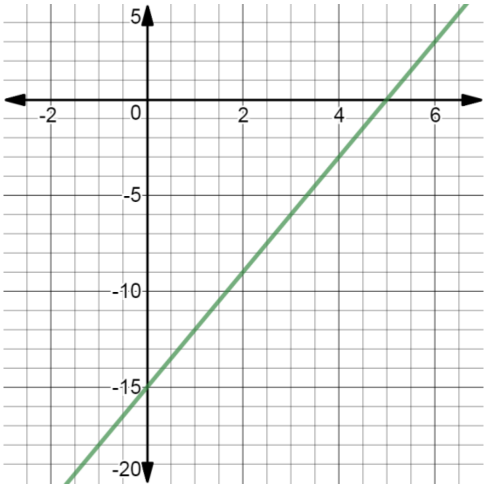

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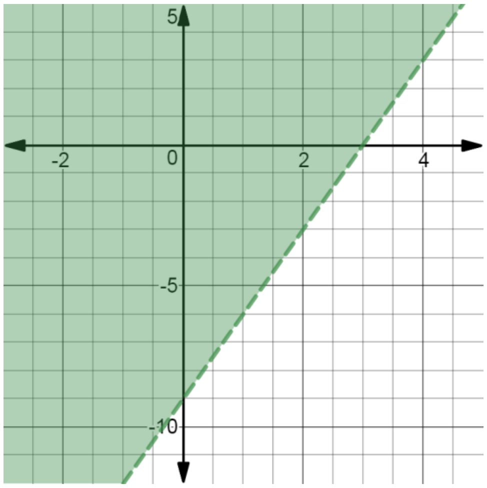

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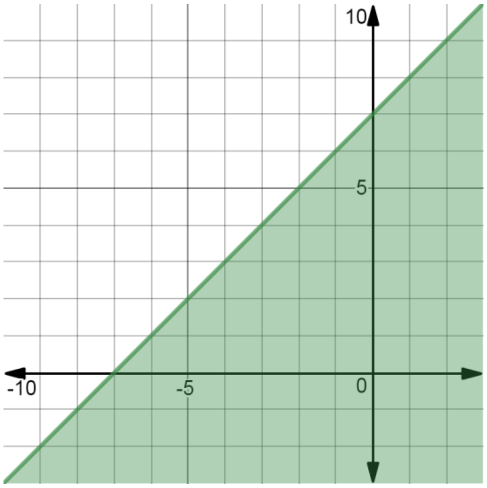

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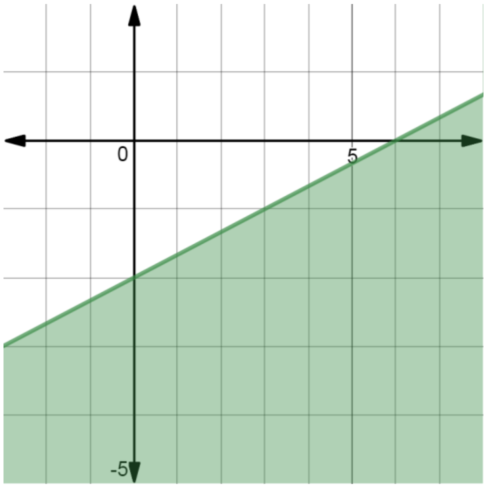

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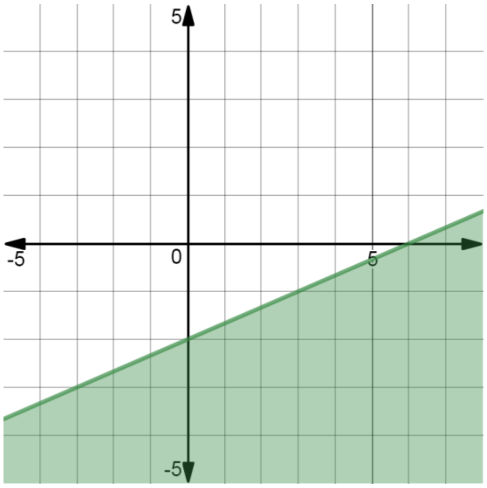

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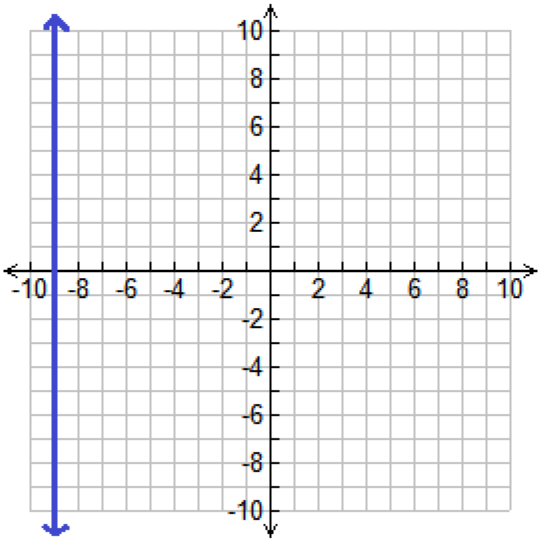

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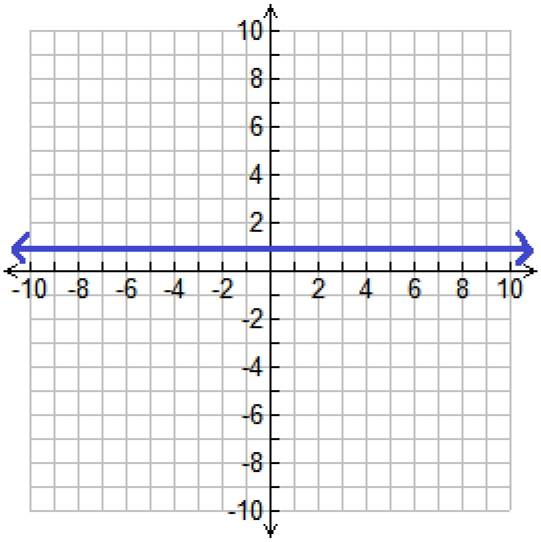


Section 4.2 Linear Programming – The Graphical Method

Linear programming gained prominence in World War II with a variety of problems, including fuel related problems. Gasoline and oil were precious commodities during war. Conserving these resources is a priority. For example, suppose we have 20 munitions storage locations and 30 requests for munitions from different locations. One possible linear programming problem is this: how do we fill all 30 of the requests (all deliveries are by truck) while at the same time using a minimum amount of fuel?

The problem above is long and involved. In fact, its solution was a factor leading to the development of the computer. We illustrate linear programming problems in detail with a simpler example.

Example 1: Shipping Cargo

A truck traveling from California to Oregon is to be loaded with two types of cargo. Each crate of cargo P is 4 cubic feet in volume, weighs 100 pounds, and earns $12 for the driver. Each crate of cargo Q is 3 cubic feet in volume, weighs 200 pounds, and earns $11 for the driver. The truck can carry no more than 300 cubic feet of crates, and no more than 10,000 pounds of cargo. In addition, the number of crates of cargo Q must be less than or equal to twice the number of crates of cargo P. How many crates of each type of cargo should the driver haul to maximize earnings?

Solution:

First, summarize the given information in a table. Include variables and totals in the table.

Type of Crate	Volume	Weight	Earnings Per Crate	Number of Crates
P	4 cubic feet	100 pounds	$12	x
Q	3 cubic feet	200 pounds	$11	y
Capacity	≤ 300 cubic feet	≤ 10,000 pounds		

The last sentence of the problems tells us how to define our variables by telling us what to find. Let x represent the number of crates of cargo P and y represent the number of crates of cargo Q.

A **constraint** is a condition imposed on a problem. Constraints in linear programming problems are stated as inequalities. The constraint in the second to last sentence is turned into an inequality by writing and translating:

$$\begin{array}{rcl} \text{number of crates of cargo Q} & \leq & \text{twice the number of crates of cargo P} \\ y & \leq & 2 \cdot x \end{array}$$

Second, build the constraints and the objective function. A constraint is a condition imposed upon the variables in a problem. It is expressed as an inequality, using the symbols <, ≤, >, or

$\geq$. In a table such as the previous, either the rows or columns (not both) will give us some of the constraints. The constraints for this problem are:

volume constraint $4x+3y \leq 300$
weight constraint $100x+200y \leq 10,000$

crates constraint $y \leq 2x$
nonnegative variables constraint $x \geq 0, y \geq 0$

Each constraint reads as a condition. For example, look at the volume constraint. Since one crate of cargo P occupies 4 cubic feet of space in the truck (two crates occupy $2 \cdot 4 = 8$ cubic feet, three crates occupy $3 \cdot 4 = 12$ cubic feet, and so on), then *x* crates occupy $x \cdot 4 = 4x$ cubic feet of space. Similarly, since each crate of cargo Q occupies 3 cubic feet of space, *y* crates occupy $3y$ cubic feet of space in the truck. Since the truck cannot fit any more than 300 cubic feet of cargo, we have the condition $4x+3y \leq 300$.

(*Aside*: Hence, you see a truism in mathematics. Mathematics reads just like English, but it is written in symbols. All of mathematics is this way. And this is wherein its challenge lies. Math is shorthand and the challenge is to get used to deciphering it.)

The conditions $x \geq 0$ and $y \geq 0$ are standard in linear programming applications. They are common sense. Try translating them into meaningful English sentences before reading the next sentence. Since *x* and *y* represent the number of crates of cargo, and we cannot load a negative number of crates into the truck, then *x* and *y* must be 0 (loads no crates into the truck) or positive. These conditions will make your linear programming life easier because they restrict the graph we draw to the first quadrant.

The earnings column in the table <u>is not</u> a constraint. Since the last sentence in the problem says to maximize earnings, this column gives us an equation. This earnings equation is called our objective function. We write it as:

$$E = \$12x + \$11y$$

Our objective is to find the number of crates of cargo *x* and *y* that maximize the trucker's earnings *E* subject to the five constraints.

Third, graph all of the constraints to find the feasible solution set. This step is the most work. I want to find all ordered pairs (x, y) that satisfy all five of the constraints (conditions) in this problem. There are an infinite number of such ordered pairs, represented by a shaded region on the graph. Then we want to sift through this infinity of points and find the single ordered pair that maximizes the earnings. This sounds like finding a needle in a haystack. Fret not! Linear programming theory provides us a shortcut. Let's begin to graph.

$\boxed{\text{Graph the constraints } x \geq 0 \text{ and } y \geq 0:}$ These two constrains restrict our graph to the first quadrant. So begin by drawing the first quadrants. We will graph the other three constraints by finding the *x*- and *y*-intercepts, as done in the previous section. See the graph.

$\boxed{\text{Graph } 4x+3y \le 300:}$ We first graph the equation $4x+3y=300$. Find two points satisfying the equation. Points with 0 coordinates are easiest to work with. We called this the intercept-method in the previous section. We put our results in a small table of values below.

x	***y***	(x,y)
0	$4(0)+3y=300$ $3y=300$ $y=100$	(0,100)
$4x+3(0)=300$ $4x=300$ $x=75$	0	$(75,0)$

Point the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $4x+3y \le 300$ (the "equals" in the symbol $\le$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ because it is easy to work with: $4(0)+3(0) \le 300$. This results in $0 \le 300$, which is a true statement. This means that every point on the same side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the southwestern part of the graph (Note: Write the equation of the line on its graph. We will label each line with the equation and use this labeling later.)

$\boxed{\text{Graph } 100x+200y \le 10{,}000:}$ Now we graph the equation $100x+200y=10{,}000$ using the same values for *x* and *y*.

x	***y***	(x,y)
0	$100(0)+200y=10{,}000$ $200y=10{,}000$ $y=50$	$(0,50)$
$100x+200(0)=10{,}000$ $100x=10{,}000$ $x=100$	0	$(100,0)$

Point the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $100x+200y \le 10{,}000$ (again, the "equals" in the symbol $\le$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ again because it is easy to work with: $100(0)+200(0) \le 10{,}000$. This results in $0 \le 10{,}000$, which is a true statement. This means that every point on the same side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the southwestern part of the graph. Don't forget to write the equation of the line on its graph.

$\boxed{\text{Graph } y \le 2x:}$ Now we graph the equation $y = 2x$ using the same values for *x* and *y*.

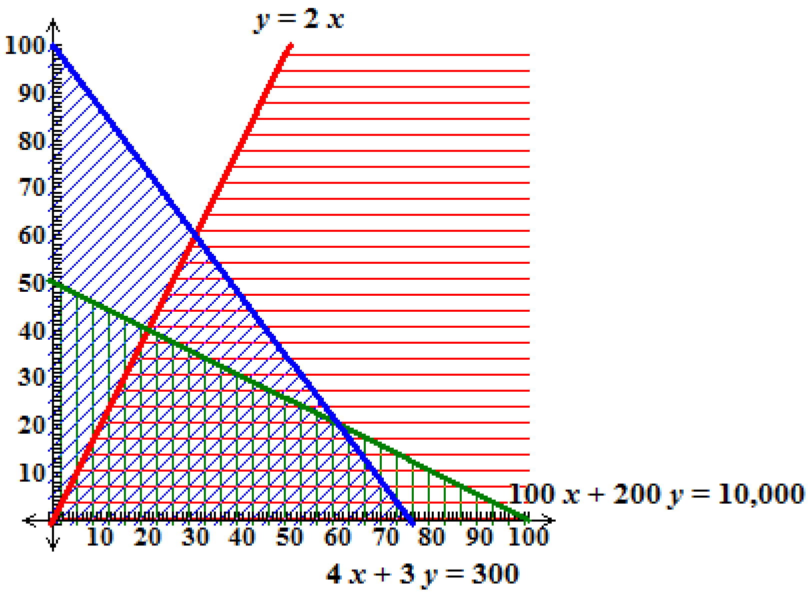
x	***y***	(x, y)
0	$y = 2(0)$ $y = 0$	$(0, 0)$
$0 = 2x$ $0 = x$	0	$(0, 0)$

Notice the *x*- and *y*-intercept are the same, namely the origin. Therefore, we need another point. Choose a random, reasonable number for *x* and solve for *y*.

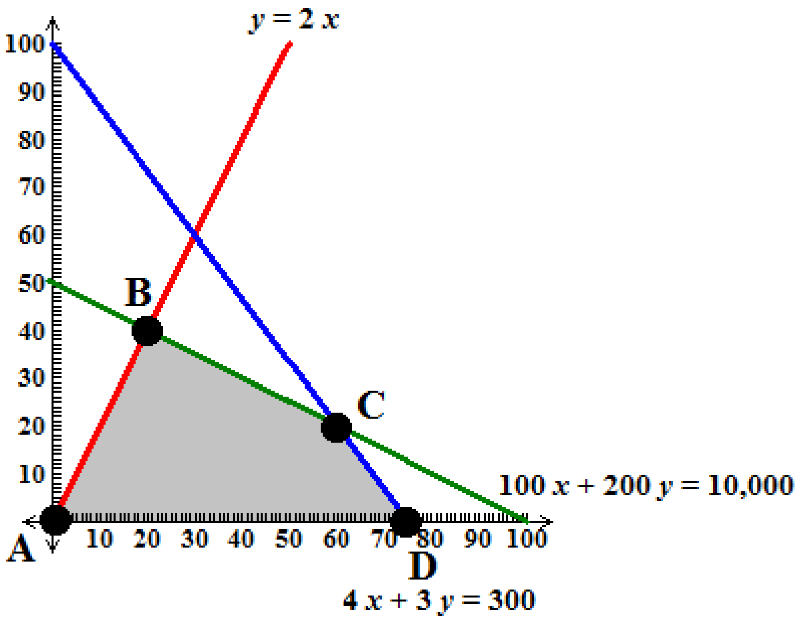
x	***y***	(x, y)
40	$y = 2(40)$ $y = 80$	$(40, 80)$

Point the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $y \le 2x$ (again, the "equals" in the symbol $\le$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. We cannot test $(0, 0)$ since it is on the line, so let's test $(0,10)$: $10 \le 2(0)$. This results in $10 \le 0$, which is a false statement. This means that every point on the same side of the line as $(0,10)$ does not satisfy the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northwestern part of the graph. Don't forget to write the equation of the line on its graph.

When constructing the graph, you may wish to use colored pencils or a creative shading style. Ultimately, we want to find the region of the graph where all shaded areas overlap. On the graph, we used red horizontal lines when shading $y \le 2x$, blue diagonal lines when shading $4x + 3y \le 300$, and green vertical lines when shading $100x + 200y \le 10,000$.



Now what?! The feasible solution set is the region all shaded areas overlap. It is the set of all points (x, y) that satisfy all five of our constraints simultaneously. We must find the single point (in the infinite number of points in the feasible solution set) that gives the trucker the greatest earnings. Fortunately for us, the **Fundamental Principle of Linear Programming** tells us that this point lies on the boundary of the feasible set. In fact, it is one of the four **vertex points**, also called **corner points**, labeled A, B, C, and D on the graph:



In other words, the **Fundamental Principle of Linear Programming** states that the ordered pair that maximizes of minimizes the objective function is a vertex on the boundary of the feasible set. The reason we graph the feasible set is to get the vertex points. Now we need to find the ordered pairs, (x, y), of vertices A, B, C, and D. Vertices A and D are easy to identify based on the graph and are $(0,0)$ and $(75,0)$, respectively. Vertices B and C must be determine by solving a system of equations. Don't attempt to guess the points based on the graph.

We want to find the intersection point of two lines. We will use a combination of the substitution and addition/elimination methods (A review of the substitution method can be found here and a review of the addition/elimination method can be found here).

Determining Vertex B: Vertex B is located at the intersection of the lines $y = 2x$ and $100x + 200y = 10,000$. Using the substitution method, we can substitute $2x$ in the place of y in the second equation:

$$\begin{aligned}&100x + 200(2x) = 10,000\\&100x + 400x = 10,000\\&500x = 10,000\\&x = 20\end{aligned}$$

If $x = 20$, substitute this for x into either equation:

$$\begin{aligned}&y = 2x\\&y = 2(20)\\&y = 40\end{aligned}$$

Therefore, vertex B is located at (20,40).

Determining Vertex C: Vertex C is located at the intersection of the lines $100x + 200y = 10,000$ and $4x + 3y = 300$. It may be more convenient to solve this system using the addition/elimination method. With this method, we want the coefficients of x or y to be opposites. One way to achieve this is to multiple each term of the second equation by -25. This gives the following system of equations.

$$\begin{cases}100x + 200y = 10,000\\-100x - 75y = -7,500\end{cases}$$

We now add these two equations together, term by term, to get $125y = 2,500$ or $y = 20$. Substitute $y = 20$ into either of the two original equations to solve for x:

$$\begin{aligned}&4x + 3(20) = 300\\&4x + 60 = 300\\&4x = 240\\&x = 60\end{aligned}$$

Therefore, vertex C is located at (60,20). Now we determine which of the four vertices optimizes (in this case, maximizes) the objective function $E = 12x + 11y$.

Vertex	$E = 12x + 11y$
A (0,0)	$E = 12(0) + 11(0) = \$0$
B (20,40)	$E = 12(20) + 11(40) = 240 + 440 = \$680$
C (60,20)	$E = 12(60) + 11(20) = 720 + 220 = \$940$
D (75,0)	$E = 12(75) + 11(0) = 900 + 0 = \$900$

Recall that *x* represents the number of crates P to load onto the truck and *y* represents the number of crates Q. Based on the table, vertex C satisfies all five of the constraints and gives us maximum earnings. In conclusion, the truck driver will make a maximum earning of \$940 if 60 crates of cargo P and 20 crates of cargo Q are hauled.

Try it Now 1

The AB Lacrosse Company manufactures two types of lacrosse sticks. Stick P requires 2 labor-hours for cutting, 1 labor-hour for stringing, and 2 labor-hours for finishing, and is sold for a profit of \$12. Stick Q requires 1 labor-hour for cutting, 2 labor-hours for stringing, and 2 labor-hours for finishing, and is sold for a profit of \$15. Each week the company has available 660 labor-hours for cutting, 680 labor-hours for stringing, and 750 labor-hours for finishing. How many lacrosse sticks of each type should be manufactured each week to maximize profits?

Example 2

Let's examine another example, where the linear programming model is already developed.

$$\text{Minimize } C = 3x + 9y \quad \text{subject to} \quad \begin{array}{l} x + y \geq 35 \\ 2x + 5y \geq 100 \\ x \geq 0 \\ y \geq 0 \end{array}.$$

Solution:

$\boxed{\text{Graph the constraints } x \geq 0 \text{ and } y \geq 0:}$ These two constrains restrict our graph to the first quadrant. So begin by drawing the first quadrants. We will graph the other three constraints by finding the *x*- and *y*-intercepts, as done in the previous problem. See the graph.

$\boxed{\text{Graph } x + y \geq 35:}$ We first graph the equation $x + y = 35$. Find two points satisfying the equation. Points with 0 coordinates are easiest to work with. We called this the intercept-method in the previous section. We put our results in a small table of values.

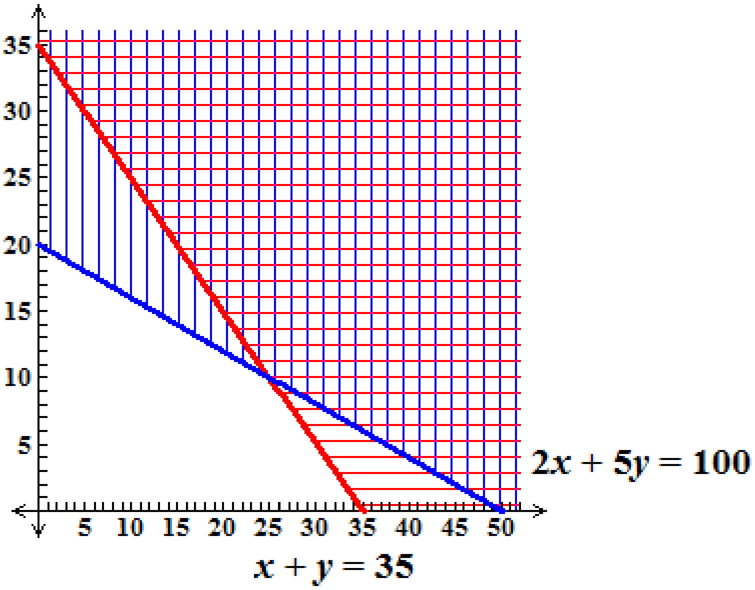
x	**y**	(x, y)
0	$(0)+y=35$ $y=35$	$(0,35)$
$x+(0)=35$ $x=35$	0	$(35,0)$

Point the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $x+y \ge 35$ (the “equals” in the symbol $\le$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ because it is easy to work with: $(0)+(0) \ge 35$. This results in $0 \ge 35$, which is a false statement. This means that every point on the other side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northeastern part of the graph. (Note: Write the equation of the line on its graph. We will label each line with the equation and use this labeling later.)

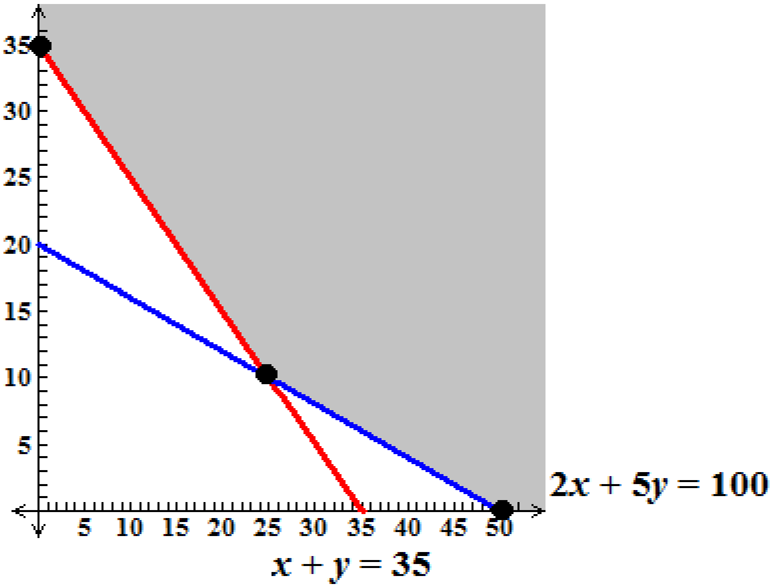
Graph $2x+5y \ge 100$: Now we graph the equation $2x+5y=100$ using the same values for *x* and *y*.

x	**y**	(x, y)
0	$2(0)+5y=100$ $5y=100$ $y=20$	$(0,20)$
$2x+5(0)=100$ $2x=100$ $x=50$	0	$(50,0)$

Point the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $2x+5y \ge 100$ (again, the “equals” in the symbol $\le$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ again because it is easy to work with: $2(0)+5(0) \ge 100$. This results in $0 \ge 100$, which is a false statement. This means that every point on the other side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northeastern part of the graph. Don’t forget to write the equation of the line on its graph.



When constructing the graph, you may wish to use colored pencils or a creative shading style. Ultimately, we want to find the region of the graph where all shaded areas overlap. On the graph, we used red horizontal lines when shading $x + y \geq 35$ and blue vertical lines when shading $2x + 5y \geq 100$. The area where all regions overlap is the northeastern part of the graph and this establishes three vertices.



We have three vertices. Two vertices are obvious: $(0,35)$ and $(50,0)$. The third is located at the intersection of the lines $x + y = 35$ and $2x + 5y = 100$. It may be more convenient to solve this system using the addition/elimination method. With this method, we want the coefficients of x or y to be opposites. One way to achieve this is to multiple each term of the first equation by -2. This gives the following system of equations.

$$\begin{cases} -2x-2y=-70 \\ 2x+5y=100 \end{cases}$$

We now add these two equations together, term by term, to get $3y=30$ or $y=10$. Plug $y=10$ into either of the two original equations to solve for *x*:

$$\begin{aligned} &x+y=35 \\ &x+(10)=35 \\ &x=25 \end{aligned}$$

Therefore, the third vertex is located at $(25,10)$. Now we determine which of the three vertices optimizes (in this case, minimizes) the objective function $C=3x+9y$.

Vertex	$C=3x+9y$
$(0,35)$	$C=3(0)+9(35)=315$
$(50,0)$	$C=3(50)+9(0)=150$
$(25,10)$	$C=3(25)+9(10)=165$

The minimum value, $C=150$, is attained when $x=50$ and $y=0$.

Try it Now 2

Minimize $W=8x+4y$ subject to

$$\begin{aligned} &x+2y\geq 56 \\ &3x+4y\geq 120 \\ &x\geq 0 \\ &y\geq 0 \end{aligned}$$

Although this method tests your graphing skills, it is efficient for solving simpler linear programming models in two-variables. This method is no longer practical when working with many equations or with equations in three or more variables. In the next section, we will examine a non-graphical approach for deriving solutions to a linear programming model.

Try it Now Answers

1. The maximum profit of $5,415 is attained when 70 units of stick P and 305 units of stick Q are produced.
2. The minimum value, $W=120$, is attained when $x=0$ and $y=30$.

Section 4.2 Exercises

Graph the feasible region for each set of inequalities. Find the vertex points.

1. $x+2y\le 4$
   $2x-y\le 3$
   $x\ge 0$
   $y\ge 0$

2. $x-y\le 3$
   $x+y\le 5$
   $x\ge 0$
   $y\ge 0$

Solve each linear programming programing by graphing and then determining which vertex maximizes or minimizes the objective function.

3. Maximize $M=4x+2y$ subject to
   $2x+3y\le 24$
   $x+6y\le 30$
   $x\ge 0$
   $y\ge 0$

4. Maximize $M=7x+9y$ subject to
   $x+2y\le 10$
   $4x+y\le 12$
   $x\ge 0$
   $y\ge 0$

5. Minimize $P=2x+3y$ subject to
   $3x+y\ge 18$
   $x+5y\ge 20$
   $x\ge 0$
   $y\ge 0$

6. Minimize $C=10x+13y$ subject to
   $x+y\ge 13$
   $10x+3y\ge 60$
   $x\ge 0$
   $y\ge 0$

Solve each application problem.

7. The Toys Galore Company makes baseball and football games. Its manufacturing process has the following labor constraints. Each baseball game requires 2 hours of assembly and 2 hours of testing. Each football game requires 3 hours of assembly and 1 hour of testing. Each day there are 48 hours available for assembly and 32 hours available for testing. How many of each game should Toys Galore make each day to maximize the total number of games produced daily?

8. Sunny Contractors builds two types of homes. The rancher requires one lot, $25,000 in capital, and 160 man-days to build, and is sold for a profit of $4,800. The colonial requires two lots, $40,000 in capital, and 200 man-days to build, and is sold for a profit of $5,800. The contractor owns 200 lots, and has $4,600,000 in capital and 28,000 man-days of labor. If the company is certain all homes will sell, how many homes of each type should Sunny Contractors build in order to maximize its profit?

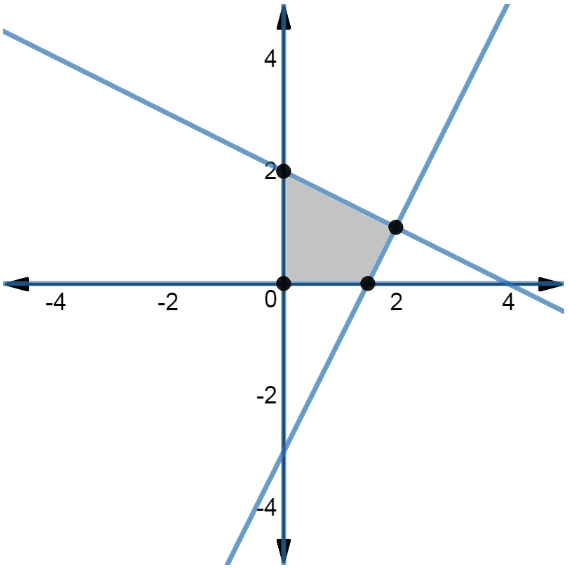
9. Coal Inc. owns two mines. On each day of the week the Pennsylvania mine produces 8 tons of anthracite (hard coal), 10 tons of bituminous (semi-hard coal), and 12 tons of lignite (soft coal). On each day of the week the Ohio mine produces 12 tons of

anthracite, 8 tons of bituminous, and 6 tons of lignite. It costs the company \$3,000 per day to operate the Pennsylvania mine and \$3,800 per day to operate the Ohio mine. Coal Inc. receives an order for 144 tons of anthracite, 152 tons of bituminous, and 120 tons of lignite. How many days do we operate each mine so that we fill the order while keeping the cost to a minimum?

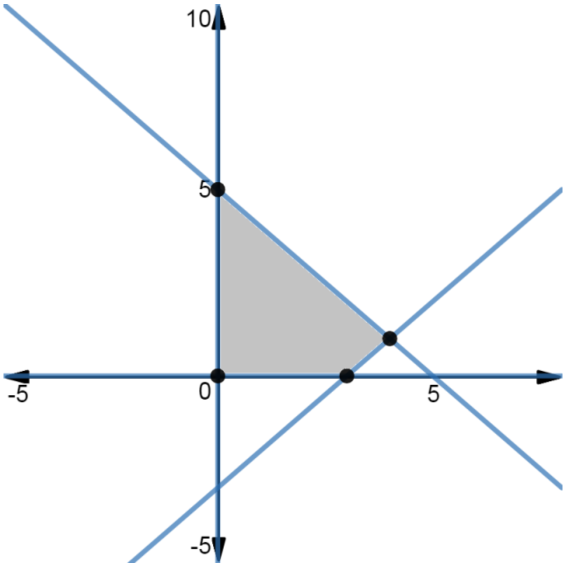
10. A company produces music players in factories in Baltimore and Frederick. Each week the Baltimore factory produces 800 economy music players and 500 premium music players. Each week the Frederick factory produces 500 economy music players and 500 premium music players. The company receives an order from an electronics store for 30,000 of the economy music players and 24,000 of the premium music players. It costs the Baltimore factory \$24,000 per week to operate and the Frederick factory \$20,000 per week to operate. How many weeks should each factory operate to fill the store order at the least cost?

Section 4.2 Exercises – Answer Key

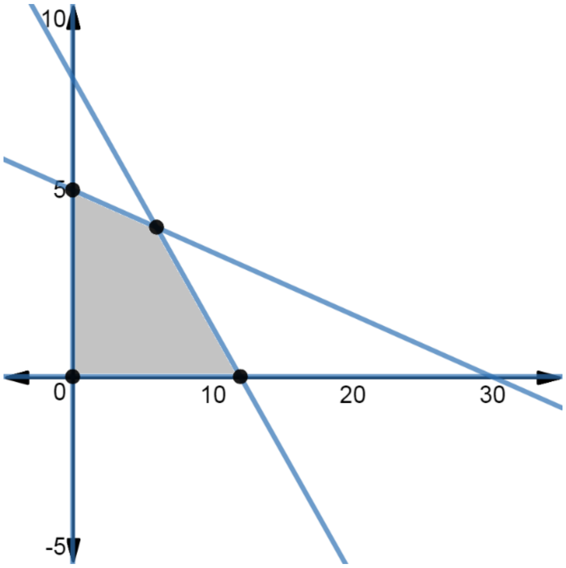
1. Vertices: $(0,0),\left(\frac{3}{2},0\right),(0,2),(2,1)$



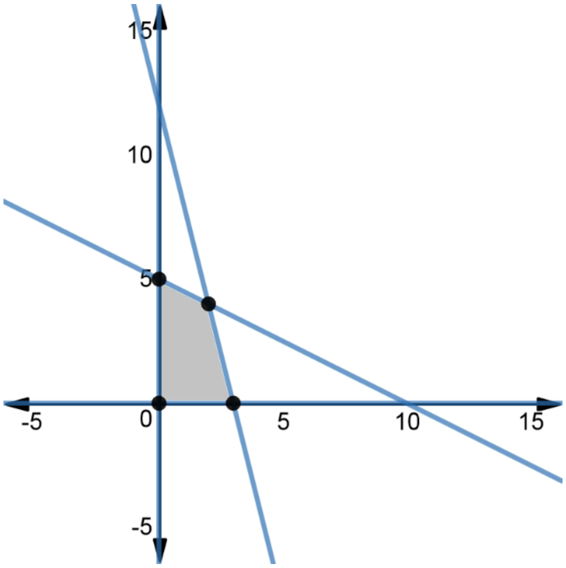
2. Vertices: $(0,0),(3,0),(0,5),(4,1)$



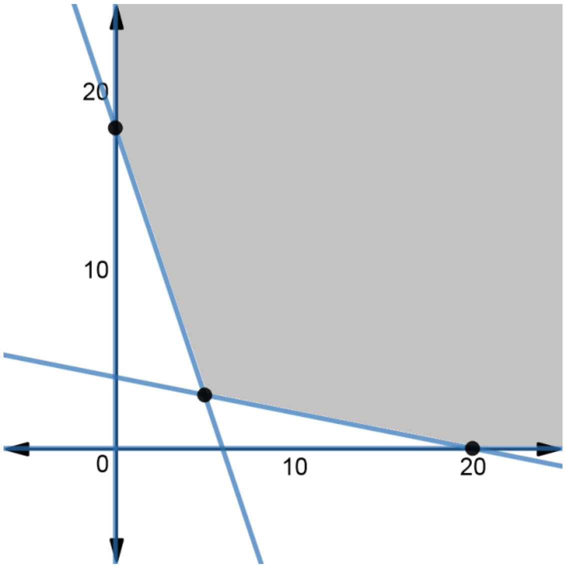
3. $M=48, x=12, y=0$



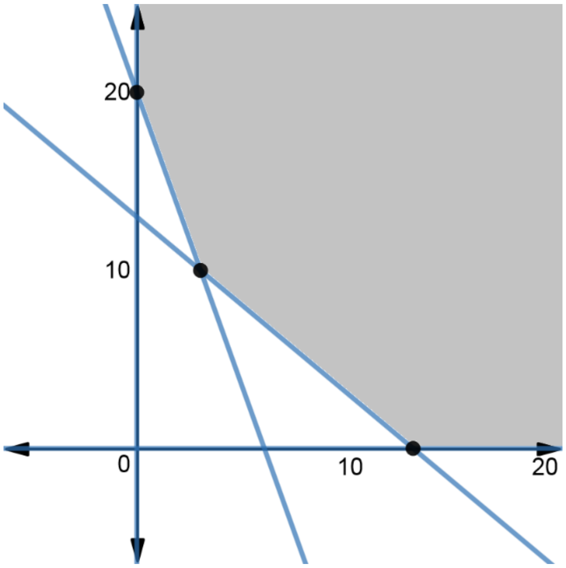
4. $M = 50, x = 2, y = 4$



5. $P = 19, x = 5, y = 3$



6. $C = 130, x = 13, y = 0$



7. Maximum production of 20 games is attained when 12 baseball and 8 football games are produced.
8. Maximum profit of $840,000 is attained when 175 ranchers and 0 colonials are built.
9. Minimum cost of $51,200 is attained when the Pennsylvania mine operates for 12 days and the Ohio mine operates for 4 days.
10. Minimum cost of $1,040,000 is attained when the Baltimore factory operates 20 weeks and the Frederick factory operates 28 weeks.

Section 4.3: Linear Programming – The Simplex Method

World View Note: George Dantzig invented the field of linear programming and it revolutionized the way government and private enterprise conducted business. In 1947, he invented the simplex method to efficiently find the optimal solution for linear programming problems.

The **simplex method** is an alternate method to graphing that can be used to solve linear programming problems—particularly those with more than two variables. We first list the algorithm for the simplex method, and then we examine a few examples.

1. Setup the problem. That is, write the objectives functions and constraints.
2. Convert the inequalities into equations. This is done by adding one *slack variable* to each inequality. Set the objective function equal to zero.
3. Construct the initial simplex tableau. Write the objective function as the bottom row.
4. We will pivot, as done in the Gauss Jordan method. To determine the pivot column, identify the most negative entry in the bottom row.
5. To identify the pivot row, calculate the quotients of the far right column and those in the pivot column (excluding the bottom row for the objective function). The smallest positive quotient identifies the pivot row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element. A quotient that is zero, negative, or undefined is not considered.
6. Perform pivoting to make this pivot element a one and all other entries in this column zero. This is done the same way as we did with the Gauss-Jordan method.
7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.
8. Extract the solution. The optimal solution is in the bottom right corner. Variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. For columns that have a 1 and all other entries 0, the value of the variable is in the far right column.

Example 1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours per week. She has determined that for every hour she works at Job I, she needs 2 hours or preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution:

In solving this problem, we will use the algorithm list above.

(Step 1) Setup the problem. That is, write the objective functions and constraints. Therefore, let x represent the number of hours per week Niki will work at Job I and y be the number of hours per week Niki will work at Job II. We are trying to maximize income, so we will label the variable to be maximized as I.

$$\text{Maximize } I = 40x + 30y \text{ subject to}$$

$$\begin{aligned} & x + y \le 12 \\ & 2x + 1y \le 16 \\ & x \ge 0 \\ & y \ge 0 \end{aligned}$$

(Step 2) Convert the inequalities into equations. This is done by adding one slack variable for each inequality. For example, to convert the inequality $x + y \le 12$ into an equation, we add a non-negative variable (usually *s*, *t*, or *u*) and we get:

$$x + y + s = 12$$

Here the variable *s* picks up the slack, and it represents the amount by which $x + y$ falls short of 12. (For example, if Niki works fewer than 12 hours, say 10, then *s* is 2.) Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

$$2x + y + t = 16$$

We will set the objective function equal to 0 by subtracting 40*x* and 30*y* from both sides. We get:

$$I - 40x - 30y = 0$$

(Step 3) We are now ready to setup the initial simplex tableau. The objective function is represented in the bottom row. The initial simplex tableau follows.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

(Step 4) The most negative entry in the bottom row identifies the pivot column.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0
↑					

-40 is the most negative entry, so it identifies the pivot column.

(Step 5) Calculate the quotients of the final column and pivot column. The smallest positive quotient identifies the pivot row. Quotients that are zero, negative, or undefined are not considered.

x	*y*	*s*	*t*	*I*		
1	1	1	0	0	12	$12/1=12$
2	1	0	1	0	16	← $16/2=8$
-40	-30	0	0	1	0	
↑						

The smallest positive quotient is 8, so this identifies the pivot row. Therefore, the pivot entry is 2.

(Step 6) Begin pivoting as done in the Gauss-Jordan method. The pivot entry becomes 1 and all entries above and below become 0. We will start by dividing R2 by 2 so the pivot entry becomes 1.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To get zeroes above and below the pivot entry, we will replace R1 with R1 – R2 and we will replace R3 with R3 + 40R2.

x	*y*	*s*	*t*	*I*	
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

(Step 7) Since a negative number still exists in the bottom row, we repeat steps 4-6.

-10 is the most negative number in the bottom row, so it identifies the pivot column. Calculating the quotients of the final column and pivot column gives 8 and 16. Since 8 is the smallest positive quotient, it identifies the pivot row.

x	*y*	*s*	*t*	*I*		
0	1/2	1	-1/2	0	4	← $4/(1/2)=8$
1	1/2	0	1/2	0	8	$8/(1/2)=16$
0	-10	0	20	1	320	
	↑					

Therefore, 1/2 is the pivot entry. We need to convert this to 1 and all entries below to 0.

x	*y*	*s*	*t*	*I*	
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

To change the pivot entry to 1, we will multiply R1 by 2/1, or simply 2.

x	y	s	t	I	
0	1	2	-1	0	8
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

To get zeroes below the pivot entry, we will replace R2 with R2 – $\frac{1}{2}$R1 and we will replace R3 with R3 + 10R1.

x	y	s	t	I	
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

We no longer have negative entries in the final row! Now step 7 is complete. If a negative number was present in the bottom row, we would repeat steps 4-7 again.

(Step 8) Extract the solution. Look for columns that have a 1 and all other entries 0. Our solution is $y = 8$, $x = 4$, and $I = 400$. All other variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. Therefore, $s = 0$ and $t = 0$. This means that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to $400.

Example 2

Solve using the simplex method.

$$\text{Maximize } Z = x + 2y + z \text{ subject to } \begin{array}{l} x + y \le 16 \\ x + 3z \le 36 \\ 5y + z \le 100 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array}$$

Solution:

(Step 1) This problem is already setup for us.

(Step 2) Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

$$\begin{array}{l} x + y + s = 16 \\ x + 3z + t = 36 \\ 5y + z + u = 100 \end{array}$$

Next, we set the objective function equal to 0.

$$Z - x - 2y - z = 0$$

(Step 3) We are now ready to setup the initial simplex tableau. The objective function is represented in the bottom row. The initial simplex tableau follows.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
0	5	1	0	0	1	0	100
-1	-2	-1	0	0	0	1	0

(Step 4) The most negative entry in the bottom row identifies the pivot column.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
0	5	1	0	0	1	0	100
-1	-2	-1	0	0	0	1	0
	↑						

-2 is the most negative entry, so it identifies the pivot column.

(Step 5) Calculate the quotients of the final column and pivot column. The smallest positive quotient identifies the pivot row. Quotients that are zero, negative, or undefined are not considered.

x	y	z	s	t	u	Z		
1	1	0	1	0	0	0	16	←16/1=16
1	0	3	0	1	0	0	36	36/0 undefined
0	5	1	0	0	1	0	100	100/5=20
-1	-2	-1	0	0	0	1	0	
	↑							

The smallest positive quotient is 16, so this identifies the pivot row. Therefore, the pivot entry is 1.

(Step 6) Begin pivoting as done in the Gauss-Jordan method. The pivot entry becomes 1 and all entries above and below become 0. Since the pivot entry is already 1, we work on making the entries below 0.

We will replace R3 with R3 – 5R1, and we will replace R4 with R4 + 2R1.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

(Step 7) Since a negative number still exists in the bottom row, we repeat steps 4-6.

-1 is the most negative number (and only negative number) in the bottom row, so it identifies the pivot column. Calculating the quotients of the final column and pivot column gives 12 and 20. Since 12 is the smallest positive quotient, it identifies the pivot row.

x	y	z	s	t	u	Z		
1	1	0	1	0	0	0	16	16/0 undefined
1	0	3	0	1	0	0	36	← 36/3=12
-5	0	1	-5	0	1	0	20	20/1=20
1	0	-1	2	0	0	1	32	
		↑						

Therefore, 3 is the pivot entry. We need to convert this to 1 and all entries below to 0.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

To change the pivot entry to 1, we will multiply R1 by 1/3 (or divide by 3).

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1/3	0	1	0	1/3	0	0	12
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

We already have 0 above the pivot entry. To get zeroes below the pivot entry, we will replace R3 with R3 –R2 and we will replace R4 with R4 + R2.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1/3	0	1	0	1/3	0	0	12
-16/3	0	0	-5	-1/3	1	0	8
4/3	0	0	2	1/3	0	1	44

We no longer have negative entries in the final row! Now step 7 is complete. If a negative number was present in the bottom row, we would repeat steps 4-7 again.

(Step 8) Extract the solution. Look for columns that have a 1 and all other entries 0. This gives $y = 16$, $z = 12$, $u = 8$, and $Z = 44$. All other variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. Therefore, $x = 0$, $s = 0$, and $t = 0$. For our model's final solution, we are only concerned about the original variables (not the slack variables).Therefore, Z attains a maximum value of 44 when $x = 0$, $y = 16$, and $z = 12$.

Try it Now 1

Solve using the simplex method.

$$\text{Maximize } R = 2x + 3y + z \text{ subject to } \begin{array}{l} x + y + z \le 40 \\ 2x + y - z \le 10 \\ x - y \le 10 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array}$$

Try it Now Answer

1. A maximum of $R = 90$ is attained when $x = 0, y = 25, z = 15$.

Section 4.3 Exercises

Solve using the simplex method.

1. Maximize $R = 2x + 4y$ subject to $\begin{array}{l} x + y \le 6 \\ x + 3y \le 12 \\ x \ge 0 \\ y \ge 0 \end{array}$

2. Maximize $R = 5x + 3y$ subject to $\begin{array}{l} x + y \le 12 \\ 2x + y \le 16 \\ x \ge 0 \\ y \ge 0 \end{array}$

3. Maximize $Z = 5x + 8y$ subject to $\begin{array}{l} x + 2y \le 30 \\ 3x + y \le 30 \\ x \ge 0 \\ y \ge 0 \end{array}$

4. Maximize $T = 2x + y + z$ subject to
$$\begin{aligned} &x + y + 2z \le 8 \\ &2x + z \le 14 \\ &x \ge 0 \\ &y \ge 0 \\ &z \ge 0 \end{aligned}$$

5. Maximize $Z = x + 2y + 3z$ subject to
$$\begin{aligned} &x + y + z \le 12 \\ &2x + y + 3z \le 18 \\ &x \ge 0 \\ &y \ge 0 \\ &z \ge 0 \end{aligned}$$

6. Maximize $P = x + 2y + z$ subject to
$$\begin{aligned} &x - 3y + z \le 3 \\ &y + z \le 10 \\ &x \ge 0 \\ &y \ge 0 \\ &z \ge 0 \end{aligned}$$

7. A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and \$20 of capital, and each acre of corn requires 16 hours of labor and \$40 of capital. The farmer has at most 800 hours of labor and \$2400 of capital available. If the profit from an acre of wheat is \$80 and from an acre of corn is \$100, how many acres of each crop should she plant to maximize her profit?

8. A factory manufactures chairs, tables, and bookcases each requiring he use of three operations: cutting, assembling, and finishing. The first operation (cutting) can be used at most 600 hours, the second (assembling) at most 500 hours, and the third (finishing) at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing. A table requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing. A bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per chair, \$30 per table, and \$25 per bookcase, how many of each item should be manufactured to maximize profit?

Section 4.3 Exercises – Answer Key

1. $R=18, x=3, y=3$
2. $R=44, x=4, y=8$
3. $Z=126, x=6, y=12$
4. $T=15, x=7, y=1, z=0$
5. $Z=27, x=0, y=9, z=3$
6. $P=53, x=33, y=10, z=0$
7. Maximum profit of $9,600 is attained when 120 acres of wheat and 0 acres of corn are planted.
8. Maximum profit of $8,500 is attained when 0 chairs, 200 tables and 100 bookcases are manufactured.

Chapter 5: Finance

Section 5.1: Interest

World View Note: The word finance came from the Old French (circa 1400) word *finance*, which embodied the idea of a debt settlement. Since 1827, the word finance has expanded to include the idea of managing money.

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$$I = Pr$$

$$A = P + I = P + Pr = P(1 + r)$$

I is the interest
A is the end amount: principal plus interest
P is the principal (starting amount)
r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

P = \$300	the principal
r = 0.03	3% rate

I = \$300(0.03) = \$9. You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = Prt$$

$$A = P + I = P + Prt = P(1+rt)$$

I is the interest
A is the end amount: principal plus interest
P is the principal (starting amount)
r is the interest rate in decimal form
t is time, in years

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, with a maturity in 4 years. How much interest will you earn?

P = $1000	the principal
r = 0.04	2% rate per half-year
t = 4	4 years

I = $1000(0.04)(4) = $160. You will earn $160 interest total over the four years.

Try it Now 1
A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,
$P = \$1000$
$r = 0.0025$ (0.25%)
$I = \$1000\ (0.0025) = \$2.50$
$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.
In the second month,
$P = \$1002.50$
$I = \$1002.50\ (0.0025) = \$2.51$ (rounded)
$A = \$1002.50 + \$2.51 = \$1005.01$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting Balance	Interest Earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73

10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_n represents the amount of money after n months, then we could write the recursive equation:

$P_0 = \$1000$
$P_n = (1+0.0025)P_{n-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$
$P_1 = 1.0025P_0 = 1.0025\ (1000)$
$P_2 = 1.0025P_1 = 1.0025\ (1.0025\ (1000)) = 1.0025^{2}(1000)$
$P_3 = 1.0025P_2 = 1.0025\ (1.0025^2(1000)) = 1.0025^3(1000)$
$P_4 = 1.0025P_3 = 1.0025\ (1.0025^3(1000)) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_n = (1.0025)^n(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year. Generalizing our result, we could write

$$P_n = P_0\left(1+\frac{r}{n}\right)^{nt}$$

In this formula:
n is the number of compounding periods (months in our example) in a year
r is the annual interest rate
t is the time in years.

Since we have previously used A as the end amount (principal plus interest), and P as the principal, we will change to that now.

Compound Interest

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A is the balance in the account after t years.
P is the starting balance of the account (also called initial deposit, or principal)
r is the annual interest rate in decimal form
n is the number of compounding periods in one year.

If the compounding is done annually (once a year), $n = 1$.

If the compounding is done semiannually, $n = 2$.
If the compounding is done quarterly, $n = 4$.
If the compounding is done monthly, $n = 12$.
If the compounding is done daily, $n = 365$.
The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

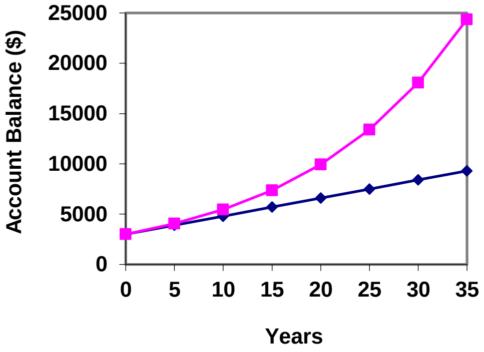
In this example,

$P = \$3000$	the initial deposit
$r = 0.06$	6% annual rate
$n = 12$	12 months in 1 year
$t = 20$	since we're looking for how much we'll have after 20 years

So $A = 3000\left(1+\frac{0.06}{12}\right)^{20\cdot 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating Exponents on the Calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5\cdot5\cdot5=125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators. Most scientific calculators have a button for exponents. It is typically either labeled like:

$\boxed{\wedge}$, $\boxed{y^x}$, or $\boxed{x^y}$.
To evaluate 1.005^{240} we'd type 1.005 $\boxed{\wedge}$ 240, or 1.005 $\boxed{y^x}$ 240. Try it out - you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example, we're looking for *P*.

$r = 0.04$	4%
$n = 4$	4 quarters in 1 year
$t = 18$	Since we know the balance in 18 years
$A = \$40{,}000$	The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for *P*.

$$40000 = P\left(1+\frac{0.04}{4}\right)^{18\cdot 4}$$

$$40000 = P(2.047099312)$$

$$P = \frac{40000}{2.047099312} = \$19539.84$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding
It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. For example, rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

$P = \$1000$	the initial deposit
$r = 0.05$	5%
$n = 12$	12 months in 1 year
$t = 30$	since we're looking for the amount after 30 years

If we first compute r/n, we find 0.05/12 = 0.00416666666667

Here is the effect of rounding this to different values:

***r/n* rounded to:**	**Gives *A* to be:**	**Error**
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your Calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$A = 1000\left(1 + \frac{0.05}{12}\right)^{12\cdot 30}$$

We can quickly calculate 12×30 = 360, giving $A = 1000\left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this:	Calculator displays:
0.05 $\boxed{\div}$ 12 $\boxed{=}$	0.00416666666667
$\boxed{+}$ 1 $\boxed{=}$	1.00416666666667
$\boxed{y^x}$ 360 $\boxed{=}$	4.46774431400613
$\boxed{\times}$ 1000 $\boxed{=}$	4467.74431400613

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 $\boxed{\times}$ (1 $\boxed{+}$ 0.05 $\boxed{\div}$ 12) $\boxed{y^x}$ 360 $\boxed{=}$

Section 5.1 Exercises

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?

2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?

3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?

4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?

5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?

6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?

7. You deposit $2000 in an account earning 3% interest compounded monthly.
   a. How much will you have in the account in 20 years?
   b. How much interest will you earn?

8. You deposit $10,000 in an account earning 4% interest compounded monthly.
   a. How much will you have in the account in 25 years?
   b. How much interest will you earn?

9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.

10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest compounded quarterly.

Section 5.1 Exercises – Answer Key

1. $210
2. $450
3. 8.16% annual rate
4. $4,975.12
5. $488.67
6. $3,869.68
7. (a) $3,641.51 (b) $1,641.51
8. (a) $27,137.65 (b) $17,137.65
9. $3,717.14
10. $16,394.93

Section 5.2: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_n = \left(1+\frac{r}{n}\right)P_{n-1}$$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

$$P_n = \left(1+\frac{r}{n}\right)P_{n-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

r = 0.06 (6%)
n = 12 (12 compounds/deposits per year)
d = $100 (our deposit per month)

Writing out the recursive equation gives

$$P_n = \left(1+\frac{0.06}{12}\right)P_{n-1} + 100 = (1.005)P_{n-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$
$$P_1 = (1.005)P_0 + 100 = 100$$
$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$
$$P_3 = (1.005)P_2 + 100 = (1.005)\left(100(1.005)+100\right) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after n deposits, we'd have saved:

$$P_n = 100(1.005)^{n-1} + 100(1.005)^{n-2} + \cdots + 100(1.005) + 100$$

In other words, after *n* months, the first deposit will have earned compound interest for *n*-1 months. The second deposit will have earned interest for *n*-2 months. Last month's deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though—it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_n = 1.005\left(100\left(1.005\right)^{n-1} + 100\left(1.005\right)^{n-2} + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_n = 100\left(1.005\right)^{n} + 100\left(1.005\right)^{n-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{lclllll} 1.005P_n & = & 100\left(1.005\right)^{n} + & 100\left(1.005\right)^{n-1} + \cdots + & 100(1.005) & \\ P_n & = & & 100\left(1.005\right)^{n-1} + \cdots + & 100(1.005) & +100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_n - P_n = 100\left(1.005\right)^{n} - 100$$

Solving for P_n

$$0.005P_n = 100\left(\left(1.005\right)^{n} - 1\right)$$

$$P_n = \frac{100\left(\left(1.005\right)^{n} - 1\right)}{0.005}$$

Replacing *n* months with 12*t*, where *t* is measured in years, gives

$$P_t = \frac{100\left(\left(1.005\right)^{12t} - 1\right)}{0.005}$$

Recall 0.005 was *r/n* and 100 was the deposit *d* (which we instead call *P* in a moment). 12 was *m*, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$A=\frac{P\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}{\left(\frac{r}{n}\right)}$$

A is the balance in the account after t years.
P is the regular deposit (the amount you deposit each year, each month, etc.)
r is the annual interest rate in decimal form.
n is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

This formula is solved for *A*, which is appropriate when we want to determine the balance in the account after a particular time period. Sometimes we may wish to determine *P*, the regular deposit needed, to reach a particular savings goal. If we solve for *P*, another version of the annuity formula follows.

$$P=\frac{A\left(\frac{r}{n}\right)}{\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}$$

For example, if the compounding frequency isn't stated:
If you make your deposits every month, use monthly compounding, $n = 12$.
If you make your deposits every year, use yearly compounding, $n = 1$.
If you make your deposits every quarter, use quarterly compounding, $n = 4$.
If you make your deposits every day, use daily compounding, $n = 365$.

When do you use this?

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit
Annuity: Many deposits.

Example 1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

In this example,

P = \$100	the monthly deposit
r = 0.06	6% annual rate
n = 12	since we’re doing monthly deposits, we’ll compound monthly
t = 20	we want the amount after 20 years

Putting this into the equation:

$$A=\frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$

$$A=\frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$A=\frac{100(3.310204476-1)}{(0.005)}$$

$$A=\frac{100(2.310204476)}{(0.005)}\approx \$46204.09$$

The account will grow to \$46,204.09 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is \$46,204.09 - \$24,000 = \$22,204.09.

Example 2

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

In this example, we’re looking for *P*.

r = 0.08	8% annual rate
n = 12	since we’re depositing monthly
t = 30	30 years
A = \$200,000	The amount we want to have in 30 years

$$P=\frac{A\left(\frac{r}{n}\right)}{\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}=\frac{200,000\left(\frac{0.08}{12}\right)}{\left(\left(1+\frac{0.08}{12}\right)^{12\cdot 30}-1\right)}=\frac{1333.333333}{\left((1.006666667)^{360}-1\right)}\approx 134.20$$

So you would need to deposit \$134.20 each month to have \$200,000 in 30 years if your account earns 8% interest

Try it Now 1

A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Try it Now Answer

1. The account will have \$21,282.07 after 10 years, with interest of \$3,032.07.

Section 5.2 Exercises

1. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would he need to deposit in the account each month?
    b. How much total money will he put into the account?
    c. How much total interest will he earn?

2. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?

3. Mike plans to retire in 15 years. He wants to make regular contributions into an annuity account so that he accumulates a total of $275,000. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the next 15 years in order to accomplish his goal?

4. Your grandparents started putting money away for you when you were born. If they put $100 each month into an account earning 3% compounded monthly, how much will be in the account when you turn 18?

5. You deposit $150 each month into an account earning 8% interest compounded monthly.

    a. How much will you have in the account in 30 years?
    b. How much total money will you put in the account?
    c. How much total interest will you earn?

6. If you put $50 into an annuity each month, and the account earns 2.75% interest compounded monthly, how much will you have in 5 years? How much is interest?

7. You would like to have $700,000 when you retire in 25 years.

    a. How much should you invest each quarter if you can earn a rate of 2.2% compounded quarterly?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?

8. Consider the following investment scenario.

    a. How much should you invest each month in order to have $600,000 if your rate of return is 4.7% compounded monthly and you want to achieve your goal in 40 years?
    b. How much interest will you earn?
    c. How much should you invest each month in order to have $600,000 if you want to achieve your goal in 20 years?
    d. If you deposit the amount you need to achieve your goal in 20 years, how much will your savings be worth after 10 years?

9. Consider the following investment scenario.

   a. How much should you invest each month in order to have $800,000 if your rate of return is 4.3% compounded monthly and you want to achieve your goal in 40 years?
   b. How much interest will you earn?
   c. How much should you invest each month in order to have $800,000 if you want to achieve your goal in 20 years?
   d. If you deposit the amount you need to achieve your goal in 20 years, how much will your savings be worth after 10 years?

Section 5.2 Exercises – Answer Key

1. (a) \$796.40 (b) \$286,704 (c) \$513,296
2. \$349.53
3. \$2,411.19
4. \$28,594.03
5. (a) \$223,553.92 (b) \$54,000 (c) \$169,553.92
6. \$3,212.10; \$212.10
7. (a) \$5,269.33 (b) \$526,933 (c) \$173,067
8. (a) \$424.99 (b) \$396,004.80 (c) \$1,510.98 (d) \$230,900.65
9. (a) \$627.64 (b) \$498,732.80 (c) \$2,108.57 (d) \$315,447.84

Section 5.3: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P=\frac{PMT\left(1-\left(1+\frac{r}{n}\right)^{-nt}\right)}{\left(\frac{r}{n}\right)}$$

P is the balance in the account at the beginning (starting amount, or principal).
PMT is the regular payment (the amount you withdraw each year, each month, etc.)
r is the annual interest rate (in decimal form. Example: 5% = 0.05)
n is the number of compounding periods in one year.
t is the number of years we plan to take withdrawals

Sometimes we wish to determine PMT given the initial balance, so this formula can be rewritten as

$$PMT=\frac{P\left(\frac{r}{n}\right)}{\left(1-\left(1+\frac{r}{n}\right)^{-nt}\right)}.$$

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit
Annuity: Many deposits.
Payout Annuity: Many withdrawals

Example 1

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

In this example,

$PMT = \$1000$	the monthly withdrawal
$r = 0.06$	6% annual rate
$n = 12$	since we're doing monthly withdrawals, we'll compound monthly
$t = 20$	since were taking withdrawals for 20 years

We're looking for *P*; how much money needs to be in the account at the beginning. Putting this into the equation:

$$P = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P = \frac{1000\left(1-(1.005)^{-240}\right)}{(0.005)}$$

$$P = \frac{1000(1-0.3020961416)}{(0.005)} \approx \$139,580.77$$

You will need to have \$139,580.77 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the <u>interest earned</u>. In this case it is \$240,000 - \$139,580.77 = \$100,419.23 in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $\boxed{\text{(-)}}$, some with $\boxed{\text{+/-}}$. The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: 1.005 $\boxed{\wedge}$ $\boxed{\text{(-)}}$ 240

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: 1.005 $\boxed{y^x}$ 240 $\boxed{\text{(-)}}$ =

Give it a try - you should get $1.005^{-240} = 0.3020961416$.

Example 2

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

In this example, we're looking for *PMT*.

$r = 0.08$	8% annual rate
$n = 12$	since we're withdrawing monthly
$t = 30$	30 years
$P = \$500,000$	we are beginning with \$500,000

In this case, we have to solve for *PMT*.

$$PMT = \frac{500000\left(\frac{0.08}{12}\right)}{\left(1-\left(1+\frac{0.08}{12}\right)^{-12\cdot 30}\right)} = \frac{500000\left(0.0066666667\right)}{\left(1-\left(1.0066666667\right)^{-360}\right)} \approx 3668.82$$

You would be able to withdraw \$3,668.82 each month for 30 years.

Try it Now 1

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Try it Now Answer

1. \$7,358.18 per year

Section 5.3 Exercises

1. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.

    a. How much do you need in your account at the beginning?
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?

2. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?

    a. How much do you need in your account at the beginning?
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?

3. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?

4. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?

5. You want to be able to withdraw the specified amount periodically from a payout annuity with the given terms. Find how much the account needs to hold to make this possible. Round your answer to the nearest dollar.

Regular withdrawal:	$1600
Interest rate:	4%
Frequency:	monthly
Time:	15 years

6. You want to be able to withdraw the specified amount periodically from a payout annuity with the given terms. Find how much the account needs to hold to make this possible. Round your answer to the nearest dollar.

Regular withdrawal:	$4500
Interest rate:	3%
Frequency:	quarterly
Time:	20 years

7. You expect to have the given amount in an account with the given terms. Find how much you can withdraw periodically in order to make the account last the specified amount of time. Round your answer to the nearest cent.

   Account balance: $550,000
   Interest rate: 2.55%
   Frequency: quarterly
   Time: 15 years

8. You expect to have the given amount in an account with the given terms. Find how much you can withdraw periodically in order to make the account last the specified amount of time. Round your answer to the nearest cent.

   Account balance: $600,000
   Interest rate: 2.75%
   Frequency: monthly
   Time: 26 years

9. You expect to have the given amount in an account with the given terms. Find how much you can withdraw periodically in order to make the account last the specified amount of time. Round your answer to the nearest cent.

   Account balance: $500,000
   Interest rate: 3.05%
   Frequency quarterly
   Time: 27 years

10. You expect to have the given amount in an account with the given terms. Find how much you can withdraw periodically in order to make the account last the specified amount of time. Round your answer to the nearest cent.

    Account balance: $550,000
    Interest rate: 4.4%
    Frequency quarterly
    Time: 20 years

Section 5.3 Exercises – Answer Key

1. (a) $320,243.29 (b) $750,000 (c) $429,756.71
2. (a) $589,088.84 (b) $1,200,000 (c) $610,911.16
3. $3582.16
4. $4,387.86
5. $216, 307
6. $269,975
7. $11,060.12
8. $2,693.93
9. $6,811.27
10. $10,373.46

Section 5.4: Loans

In the last section, you learned about payout annuities. In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P = \frac{PMT\left(1-\left(1+\frac{r}{n}\right)^{-nt}\right)}{\left(\frac{r}{n}\right)} \quad \text{or} \quad PMT = \frac{P\left(\frac{r}{n}\right)}{\left(1-\left(1+\frac{r}{n}\right)^{-nt}\right)}$$

P is the balance in the account at the beginning (the principal, or amount of the loan).
PMT is your loan payment (your monthly payment, annual payment, etc)
r is the annual interest rate in decimal form.
n is the number of compounding periods in one year.
t is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit
Annuity: Many deposits.
Payout Annuity: Many withdrawals
Loans: Many payments

Example 1

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

In this example,

PMT= \$200	the monthly loan payment
$r = 0.03$	3% annual rate
$n = 12$	since we’re doing monthly payments, we’ll compound monthly
$t = 5$	since we’re making monthly payments for 5 years

We’re looking for *P*, the starting amount of the loan.

$$P = \frac{200\left(1-\left(1+\frac{0.03}{12}\right)^{-5(12)}\right)}{\left(\frac{0.03}{12}\right)}$$

$$P = \frac{200\left(1-(1.0025)^{-60}\right)}{(0.0025)}$$

$$P = \frac{200(1-0.8608691058)}{(0.0025)} \approx \$11,130.47$$

You can afford an \$11,130.47 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you’re paying \$12,000-\$11,130.47 = \$869.53 interest total.

Example 2

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

In this example,
We’re looking for *PMT*.

$r = 0.06$	6% annual rate
$n = 12$	since we’re paying monthly
$t = 30$	30 years
P = \$140,000	the starting loan amount

In this case, we solve for *PMT*.

$$PMT = \frac{140000\left(\frac{0.06}{12}\right)}{\left(1-\left(1+\frac{0.06}{12}\right)^{-12\cdot 30}\right)} = \frac{140000(0.005)}{\left(1-(1.005)^{-12\cdot 30}\right)} \approx 839.37$$

You will make payments of \$839.37 per month for 30 years.

You’re paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of \$302,173.20 - \$140,000 = \$162,173.20 in interest over the life of the loan.

Try it Now 1

Janine bought \$3,000 of new furniture on credit. Because her credit score isn’t very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don’t already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think “how much loan will these loan payments be able to pay off in the remaining time on the loan?”

Example 3

If a mortgage at a 6% interest rate has payments of \$1,000 a month, how much will the loan balance be 10 years from the end the loan?

To determine this, we are looking for the amount of the loan that can be paid off by \$1,000 a month payments in 10 years. In other words, we’re looking for P when

PMT= \$1,000	the monthly loan payment
$r = 0.06$	6% annual rate
$n = 12$	since we’re doing monthly payments, we’ll compound monthly
$t = 10$	since we’re making monthly payments for 10 more years

$$P=\frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P=\frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)}$$

$$P=\frac{1000(1-0.5496327334)}{(0.005)}\approx \$90,073.45$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Often times answering remaining balance questions requires two steps:
1) Calculating the monthly payments on the loan
2) Calculating the remaining loan balance based on the *remaining time* on the loan

Example 4

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

First we will calculate their monthly payments.
We're looking for *PMT*.

$r = 0.04$	4% annual rate
$n = 12$	since they're paying monthly
$t = 30$	30 years
$P = \$180,000$	the starting loan amount

We set up the equation and solve for *PMT*.

$$PMT=\frac{180000\left(\frac{0.04}{12}\right)}{\left(1-\left(1+\frac{0.04}{12}\right)^{-12\cdot 30}\right)}=\frac{180000(0.0033333333)}{\left(1-(1.0033333333)^{-12\cdot 30}\right)}\approx 859.35$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$PMT = \$859.35$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$n = 12$	since they're doing monthly payments
$t = 25$	since they'd be making monthly payments for 25 more years

$$P = \frac{859.35\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$P = \frac{859.35\left(1-(1.0033333333)^{-300}\right)}{(0.0033333333)}$$

$$P = \frac{542.4214009}{(0.0033333333)} = \$162,805.99$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $162,805.99.

Over that 5 years, the couple has paid off $180,000 - $162,805.99= $17,194.01 of the loan balance. They have paid a total of $859.35 a month for 5 years (60 months), for a total of $51,561, so $51,561 - $17,194.01 = $34,366.99 of what they have paid so far has been interest.

Which equation do I use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you’re looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now Answer

1. $146.89 per month

Section 5.4 Exercises

1. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?

2. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?

3. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?

4. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What are the monthly payments? How much interest will you pay over the life of the loan?

5. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?

6. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments.
    b. How much interest will she pay over the life of the loan?

7. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate. How much does he still owe on the car?

8. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years. How much does she still owe on the mortgage?

Section 5.4 Exercises – Answer Key

1. (a) $130,397.13 (b) $252,000 (c) $121,802.87
2. (a) $12,625.50 (b) $15,000 (c) $2,374.50
3. $542.38
4. $265.61; $749.28
5. (a) Down payment of 10% is $20,000, leaving $180,000 as the loan amount
   (b) $966.28
   (c) $1,079.19 a month
6. (a) $1,618.79 (b) $312,764.40
7. $10,033.45
8. $85,258.54

Calculator Tutorials

Example	TI-83 or TI-84 Graphing Calculators	Casio fx-115 ES PLUS Scientific Calculator
Convert to a fraction or a decimal.	Sometimes you need a fraction answer but the calculator defaults to decimal. For example, if you type 2÷36 in the calculator, you'll get 0.05555555556. To convert to a fraction, press the MATH key. On the next screen, hit enter to select 1:>Frac, then hit enter once more. It should yield 1/18.	This calculator has a default setting for calculating decimals. It will automatically convert to a fraction. For example, typing 2÷36 in the calculator, in normal Math and Degree mode, yields 1/18. To convert to a decimal, press the S⇔D button, giving us 0.05555555556.
Determine standard deviation of: 5, 7, 8, 6, 12.	Enter your data by hitting the STAT key and then hitting enter to select 1: Edit. In L1, enter your data values. Use the ENTER key to move down to the next entry. Hit the STAT key again, use the > key to move to the CALC menu. Hit ENTER to select 1-Var Stats. On the next screen, both the sample and population standard deviation are given. The value you use depends on the problem and dataset. If asked for a sample standard deviation, use Sx=2.702. If asked for a population standard deviation, use σx=2.417.	First, a list must be inputted into the calculator, and then standard deviation can be taken. To enter a list, press [MODE], and option [3] to put the calculator in statistics mode. On the next screen, press [1] to select 1-VAR. Enter your data, using the = key to move to the next entry. When finished, hit the AC key. Now press SHIFT then [1]. On the next screen, select [4]. On the next screen, option [3] gives population standard deviation and option [4] gives sample standard deviation. Select [3] or [4], and then hit = on the next screen. If asked for a sample standard deviation, use sx=2.702. If asked for a population standard deviation, use σx=2.417.

<table>
<tr><th>Example</th><th>TI-83 or TI-84 Graphing Calculators</th><th>Casio fx-115 ES PLUS Scientific Calculator</th></tr>
<tr><td>Determine the correlation coefficient, r, for the data:<table><tr><td><i>x</i></td><td>3</td><td>7</td><td>9</td></tr><tr><td><i>y</i></td><td>2</td><td>12</td><td>24</td></tr></table></td><td>First, you must turn on the diagnostic setting. Hit 2nd, Catalog (0 key), scroll down to DiagnosticOn, and hit enter twice. Now enter your data by hitting the STAT key and hitting enter to select 1: Edit. In L1, enter your x values. Use the ENTER key to move down to the next entry, and use the > key to move to L2, where you enter y values. Hit the STAT key again, use the > key to move to the CALC menu. Scroll down to highlight 4:LinReg(ax+b) and select ENTER. On the next screen, you should be provided your calculated r value, approximately 0.971.</td><td>First, a list must be inputted into the calculator. To enter a list, press [MODE], and option [3] to put the calculator in statistics mode. On the next screen, press [2] to select A+BX. Enter your data, using the = key to move to the next entry. Use the < or > key to move between columns. When finished entering data, click the [AC] key. Now press SHIFT then [1]. On the next screen, select [5]. On the next screen, option [3] to select r. On the next screen, hit = to get your calculated r value, approximately 0.971.</td></tr>
<tr><td>Determine the regression equation for the data:<table><tr><td><i>x</i></td><td>3</td><td>7</td><td>9</td></tr><tr><td><i>y</i></td><td>2</td><td>12</td><td>24</td></tr></table></td><td>Repeat the steps from the previous example. On the final screen, you are given the information y=ax+b where a=3.5 and b=-9.5. Therefore, the regression equation is y=3.5x-9.5.</td><td>First, a list must be inputted into the calculator. To enter a list, press [MODE], and option [3] to put the calculator in statistics mode. On the next screen, press [2] to select A+BX. Enter your data, using the = key to move to the next entry. Use the < or > key to move between columns. When finished entering data, click the [AC] key. Now press SHIFT then [1]. On the next screen, select [5]. On the next screen, option [1] to select A. On the next screen, hit = to get the A value, -9.5. Click the [AC] key. Now press SHIFT then [1]. On the next screen, select [5]. On the next screen, option [2] to select B. On the next screen, hit = to get the B value, 3.5. The equation is in the form y=Ax+B where a=3.5 and b=-9.5. Therefore, the regression equation is y=3.5x-9.5.</td></tr>
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Example	TI-83 or TI-84 Graphing Calculators	Casio fx-115 ES PLUS Scientific Calculator
Determine P(7,4) or $_7P_4$.	Hit the 7 key and then hit the MATH key. Use the > key to select the PRB menu. Scroll down to select 2:nPr and hit ENTER. Then hit the 4 key and hit ENTER. It should provide 840.	Hit the 7 key and then press SHIFT, the × key, 4, and =. It should provide the result of 840.
Determine C(9,3) or $_9C_3$.	Hit the 9 key and then hit the MATH key. Use the > key to select the PRB menu. Scroll down to select 3:nCr and hit ENTER. Hit the 3 key and hit ENTER. It should provide the result of 84.	Hit the 9 key and then press SHIFT, the ÷ key, 3, and =. It should provide the result of 84.
If $A=\begin{pmatrix} 5 & 11 \\ -3 & 2 \end{pmatrix}$ and $B=\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, determine $A \cdot B$.	Go to the MATRIX menu by hitting 2^{nd} and then x^{-1}. Use the > key to select the EDIT menu. Select 1:[A] by hitting enter. On the next screen, enter the dimension of matrix A, which is a 2x2 matrix (hit 2, ENTER, 2, ENTER). Now type the entries: 5, ENTER, 11, ENTER, (-), 3, ENTER, 2, ENTER. Return to the MATRIX menu by hitting 2^{nd} and then x^{-1}. Use the > key to select the EDIT menu. Select 2:[B] by hitting enter. On the next screen, enter the dimension of matrix B, which is a 2x1 matrix (hit 2, ENTER, 1, ENTER). Type the entries: 4, ENTER, (-), 3, ENTER. Clear your screen by selecting 2^{nd}, Quit (MODE key). Return to the MATRIX menu by hitting 2^{nd} and then x^{-1}. Under the NAMES menu, 1:[A] 2x2 should be highlighted. Hit ENTER. Hit the × key. Return to the MATRIX menu by hitting 2^{nd} and then x^{-1}. Under the NAMES menu, scroll down to 2:[B] 2x1 and hit ENTER. Hit ENTER one more time to get the result, which is $\begin{pmatrix} -13 \\ -18 \end{pmatrix}$. Sometimes the result contains decimals but you may need the answer in fraction form. If that occurs, on this screen, then hit MATH, hit enter to select 1:>Frac, then hit enter once more.	Press[MODE], 6, then press 1 for matrix A, and press 5 for a 2x2 matrix, and enter the data. Use the = key to move to the next entry. When finished entering the data, press [MODE], 6, then press 2 for matrix B, and press 6 for a 2x1 matrix, and enter data. Press the [AC] key. To multiply, press [SHIFT], 4, then select number 3 for matrix A. Then press × for multiplication. Next press [SHIFT], 4, then select number 4 for matrix B. Press [=] for the answer, which is $\begin{pmatrix} -13 \\ -18 \end{pmatrix}$. Sometimes the answer is in decimal form but you need fractions. To see the equivalent fractions, use the arrow keys to move around the entries of the matrix. The fraction are listed on the screen.

Example	**TI-83 or TI-84 Graphing Calculators**	**Casio fx-115 ES PLUS Scientific Calculator**
Determine the inverse of the matrix $A=\begin{pmatrix} 5 & 11 \\ -3 & 2 \end{pmatrix}$.	Go to the MATRIX menu by hitting 2nd and then x^{-1}. Use the > key to select the EDIT menu. Select 1:[A] by hitting enter. On the next screen, enter the dimension of matrix A, which is a 2x2 matrix (hit 2, ENTER, 2, ENTER). Now type the entries: 5, ENTER, 11, ENTER, (-), 3, ENTER, 2, ENTER. Clear your screen by selecting 2nd, Quit (MODE key). Return to the MATRIX menu by hitting 2nd and then x^{-1}. Under the NAMES menu, 1:[A] 2x2 should be highlighted. Hit ENTER. Now select the x^{-1} key and hit ENTER. The result is provided in decimal form. For fractions, hit MATH, hit enter to select 1:>Frac, then hit enter once more. The answer is $\begin{pmatrix} 2/43 & -11/43 \\ 3/43 & 5/43 \end{pmatrix}$.	Press[MODE], 6, then press 1 for matrix A, and press 5 for a 2x2 matrix, and enter the data. Use the = key to move to the next entry. When finished entering the data, click the [AC] key. Press [SHIFT], 4, then select number 3 for matrix A. Then press the x^{-1} key and press =. The answer is $\begin{pmatrix} 2/43 & -11/43 \\ 3/43 & 5/43 \end{pmatrix}$. To see the equivalent fractions, use the arrow keys to move around the entries of the matrix. The fraction are listed on the screen.