Section 2.2 - Using Second Derivative to Find Max/Min Values & Sketch Graph

Steps A-F below are kept the same as the textbook (page 222).

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This Guide Sheet is for solving **only** this specific question (Exercise Set 2.2, problems 9-46): *"Sketch the graph. List the coordinates of where extrema or points of inflection occur. State where the function is increasing or decreasing, as well as where it is concave up or concave down."*

Original Function: $f(x) = 3x^5 - 20x^3$ ← This is Example 2 and 3 on pages 219-222 of textbook.

Step A – Find derivatives and domain.

Step A.1 – Find $f'(x)$.

$f'(x) = 15x^4 - 60x^2$

Step A.2 – Find $f''(x)$.

$f''(x) = 60x^3 - 120x$

Step A.3 – Find domain. No DNEs.

Note: Section 2.2 has mainly polynomial functions so domain is all real numbers. There is no x that does not exist (DNE) for these type of functions.

Step B – Find *critical values* and *critical points.*

Step B.1 – Find *critical values* (x-coordinate) by solving for $f'(x) = 0$ and where $f'(x) =$ DNE. Set function equal to zero, factor, then use the *Principle of Zero Products.*

$15x^4 - 60x^2 = 0$

$15x^2(x^2 - 4) = 0$

$15x^2(x+2)(x-2) = 0$

$15x^2 = 0, \quad x+2=0, \quad x-2=0$

$x = 0 \qquad x = -2 \qquad x = 2$

Note: Because the domain of $f(x)$ is all real numbers, there are no DNEs.

Step B.2 – Find corresponding y-coordinate by substituting each *critical value* (*cv*) into $f(x)$.

$f(-2) = 3(-2)^5 - 20(-2)^3 = 64$

$f(0) = 3(0)^5 - 20(0)^3 = 0$

$f(2) = 3(2)^5 - 20(2)^3 = -64$

Step B.3 – List *critical points* (x, y coordinates).

$(-2, 64)$ $(0, 0)$ $(2, -64)$ (,)

Step C – Find relative extrema and where $f(x)$ is increasing and decreasing using the Second Derivative Test (SDT).

Step C.1 – Find relative extrema by substituting each *critical value* (*cv*) from Step B.1 into $f''(x)$ to determine the sign.

$f''(-2) = 60(-2)^3 - 120(-2) = -$ relative maximum

$f''(0) = 60(0)^3 - 120(0) = 0$ failed SDT

$f''(2) = 60(2)^3 - 120(2) = +$ relative minimum

Notes:

- A *cv* is a possible relative extrema. It may not be.
- We are not concerned with the value of $f''(cv)$, only the sign.
- If sign of $f''(cv)$ is ' + ', that *cv* occurs at a *relative minimum.*
- If sign of $f''(cv)$ is ' – ', that *cv* occurs at a *relative maximum.*
- If value is *zero,* SDT fails so you must use the First Derivative Test (FDT) to determine if that *cv* is a relative extrema.
  - If you got a value of *zero,* you must **also do** Step C.4 for that *cv.*
  - If you did not get a value of *zero,* skip Step C.4.

Step C.2 – List the intervals where $f(x)$ is increasing and decreasing using the SDT results.

$x = -2$ is a relative maximum so $f(x)$ increases on $(-\infty, -2)$ and decreases on $(-2, 2)$.

$x = 2$ is a relative minimum so $f(x)$ decreases on $(-2, 2)$ and increases on $(2, \infty)$.

Notes:

- At a *relative minimum,* $f(x)$ is *decreasing* to the left of *cv* and *increasing* to its right.
- At a *relative maximum,* $f(x)$ is *increasing* to the left of *cv* and *decreasing* to its right.

Step C.3 – List relative extrema (*x*, *y* coordinates) from SDT. Get *y*-coordinates from Step B.3.

Relative Maximums: $(-2, 64)$ (,) (,)

Relative Minimums: $(2, -64)$ (,) (,)

Step C.4 – ***Do this FDT step only if*** SDT from Step C.1 resulted in a value of *zero* for a *critical value* (*cv*). Find if relative extrema exists for that *cv* using FDT.

Step C.4.1 – Write *cv* under a point on the interval line, from smallest to largest. Write the intervals adjacent to each *cv*. For each interval, choose an easy test value and substitute it into $f'(x)$ to determine the sign and find if $f(x)$ is *increasing* or *decreasing.*

Interval Line	$(-\infty, 0)$	$(0, \infty)$	(,)
Test Value	$x = -1$	$x = 1$	$x =$
Sign of $f'(x)$	$-$	$-$	
Result (increasing/ decreasing)	decreasing	decreasing	

Critical values written under the points of the interval line: 0 and an empty blank.

$f'(-1) = 15(-1)^4 - 60(-1)^2 = -$

$f'(1) = 15(1)^4 - 60(1)^2 = -$

$x = 0$ is not a relative extrema since no sign change from its left to its right.

Notes:

- We are not concerned with the value of $f'(x)$, only the sign.
- If sign of $f'(x)$ is ' – ' (*decreasing*) to the left of *cv* and ' + ' (*increasing*) to its right, this is a *relative minimum.*
- If sign of $f'(x)$ is ' + ' (*increasing*) to the left of *cv* and ' – ' (*decreasing*) to its right, this is a *relative maximum.*
- If sign of $f'(x)$ is the same to the left of *cv* and to its right, this is not a relative extrema.

Step C.4.2 – List the intervals where $f(x)$ is *increasing* and *decreasing.* List the intervals only if a relative extrema is found using the FDT results.

N/A

Notes:

- At a *relative minimum,* $f(x)$ is *decreasing* to the left of *cv* and *increasing* to its right.
- At a *relative maximum,* $f(x)$ is *increasing* to the left of *cv* and *decreasing* to its right.

Step C.4.3 – List relative extrema (x, y coordinates) from FDT. Get y-coordinates from Step B.3.

Relative Maximums: (,) (,)

Relative Minimums: (,) (,)

☑ No relative extrema found here (from FDT after SDT failed for that *cv*).

Step D – Find *Possible Points of Inflection* (PPOIs).

Step D.1 – Find x-coordinates of PPOIs by solving for $f''(x) = 0$ and where $f''(x) = \text{DNE}$. Set function equal to zero, factor, then use the *Principle of Zero Products.*

$$60x^3 - 120x = 0$$

$$60x(x^2 - 2) = 0$$

$$60x = 0$$
$$x = 0$$

$$x^2 - 2 = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$
$$x \approx \pm 1.4$$

Note: Because the domain of $f(x)$ is all real numbers, there are no DNEs.

Step D.2 – Find y-coordinates of PPOIs by substituting each PPOI x-coordinate into $f(x)$.

$$f(-\sqrt{2}) = 3(-\sqrt{2})^5 - 20(-\sqrt{2})^3 = 28\sqrt{2} \approx 39.6$$

$$f(0) = 3(0)^5 - 20(0)^3 = 0$$

$$f(\sqrt{2}) = 3(\sqrt{2})^5 - 20(\sqrt{2})^3 = -28\sqrt{2} \approx -39.6$$

Step D.3 – List PPOI *points* (x, y coordinates).

$(-\sqrt{2}, 28\sqrt{2})$ $(0, 0)$ $(\sqrt{2}, -28\sqrt{2})$ (,)

Note: Step E will determine if these PPOIs are actual *Points of Inflection* (POI). If they are not POIs, these PPOIs provide additional points to sketch the graph.

Step E – Find concavity and any POIs.

Step E.1 – Write PPOI x-coordinates (from Step D.1) under a point on the interval line, from smallest to largest. Write the intervals adjacent to each PPOI. For each interval, choose an easy test value and substitute it into $f''(x)$ to determine the sign and find concavity.

Interval Line	$(-\infty, -\sqrt{2})$	$-\sqrt{2}$ $(-\sqrt{2}, 0)$	0 $(0, \sqrt{2})$	$\sqrt{2}$ $(\sqrt{2}, \infty)$
Test Value	$x = -2$	$x = -1$	$x = 1$	$x = 2$
Sign of $f''(x)$	$-$	$+$	$-$	$+$
Result (concave up/down)	concave down	concave up	concave down	concave up

$$f''(-2) = 60(-2)^3 - 120(-2) = -$$

$$f''(-1) = 60(-1)^3 - 120(-1) = +$$

$$f''(1) = 60(1)^3 - 120(1) = -$$

$$f''(2) = 60(2)^3 - 120(2) = +$$

Notes:

- We are not concerned with the value of $f''(x)$, only the sign.
- If there is a *sign change* from the left of PPOI to its right, concavity has changed so this is a POI.
- If there is no *sign change* from the left of PPOI to its right, concavity has not changed so this is not a POI.

Step E.2 – List POI *points* (x, y coordinates). Get y-coordinates from Step D.3.

$(-\sqrt{2}, 28\sqrt{2})$ $(0, 0)$ $(\sqrt{2}, -28\sqrt{2})$ (,)

❑ No POIs.

Step F – Sketch the graph.

Step F.1 – ☑ Plot and label *critical points* from Step B.3.

Step F.2 – ☑ Sketch *short arcs* (⌢ or ⌣) to indicate concavity at relative extrema from Step C.3 and possibly Step C.4.3 (if any).

Step F.3 – ☑ Plot PPOI *points* from Step D.3.

Step F.4 – ☑ Sketch concavity (up or down) over the intervals from Step E.1.

Step F.5 – ☑ Label POI *points* from Step E.2 (if any).

Step F.6 – ☑ Plot and label additional points, if needed, to complete the graph. $x = \pm\sqrt{\frac{20}{3}}$

$y = 3x^5 - 20x^3$

$0 = 3x^5 - 20x^3$

$x^3(3x^2 - 20) = 0$

$x^3 = 0 \rightarrow x = 0$ Already have

x	*y*
2.6	0
−2.6	0

$3x^2 = 20$

$\frac{3x^2}{3} = \frac{20}{3}$

$x^2 = \frac{20}{3}$

$x = \pm\sqrt{\frac{4 \cdot 5}{3}}$

$x = \pm 2\sqrt{\frac{5}{3}}$

$x \approx \pm 2.6$

Step F.7 – ☑ Graph by plotting, labeling, and connecting points from Steps F.1 to F.6.