Trigonometric Properties & Theorems

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Chapter 6 – Trigonometric Functions

Section 6.1 – Angles and Their Measure

Degrees and Radians Conversion Formulas – pages 497 – 498

1 revolution = 2π radians $360° = 2\pi$ radians $180° = \pi$ radians

1 degree = $\frac{\pi}{180}$ radian   1 radian = $\frac{180}{\pi}$ degrees

Degree and Radian Measures of Common Angles – page 499

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees		210°	225°	240°	270°	300°	315°	330°	360°
Radians		$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Section 6.2 – Right Angle Trigonometry

Trigonometric Functions of Acute Angles – page 508

$\sin \Theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$ $\cos \Theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$ $\tan \Theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$

$\csc \Theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$ $\sec \Theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$ $\cot \Theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$

Fundamental Identities

Reciprocal Identities – page 508

$\csc \Theta = \frac{1}{\sin \Theta}$ $\sec \Theta = \frac{1}{\cos \Theta}$ $\cot \Theta = \frac{1}{\tan \Theta}$

Quotient Identities – page 508

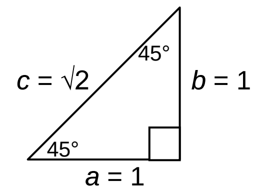
$\tan \Theta = \frac{\sin \Theta}{\cos \Theta}$ $\cot \Theta = \frac{\cos \Theta}{\sin \Theta}$

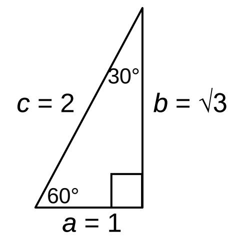
Pythagorean Identities – page 509

$\sin^2 \Theta + \cos^2 \Theta = 1$ $\tan^2 \Theta + 1 = \sec^2 \Theta$ $\cot^2 \Theta + 1 = \csc^2 \Theta$

Section 6.3 – Computing the Values of Trigonometric Functions of Acute Angles

Values of Radian/Degree Angles of the Six Trigonometric Functions – page 520							
Θ (Radians)	**Θ (Degrees)**	**sin Θ**	**cos Θ**	**tan Θ**	**csc Θ**	**sec Θ**	**cot Θ**
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$





Section 6.4 – Trigonometric Functions of General Angles

Six Trigonometric Functions of Θ – page 526

$\sin\Theta = \frac{b}{r}$ $\qquad \cos\Theta = \frac{a}{r}$ $\qquad \tan\Theta = \frac{b}{a}$

$\csc\Theta = \frac{r}{b}$ $\qquad \sec\Theta = \frac{r}{a}$ $\qquad \cot\Theta = \frac{a}{b}$

Values of Trigonometric Functions of Quadrantal Angles – page 528							
Θ (Radians)	**Θ (Degrees)**	**sin Θ**	**cos Θ**	**tan Θ**	**csc Θ**	**sec Θ**	**cot Θ**
0	0°	0	1	0	Not defined	1	Not defined
$\frac{\pi}{2}$	90°	1	0	Not defined	1	Not defined	0
π	180°	0	–1	0	Not defined	–1	Not defined
$\frac{3\pi}{2}$	270°	–1	0	Not defined	–1	Not defined	0

Signs of the Six Trigonometric Functions For Each Quadrant – page 530			
Quadrant of Θ	**sin Θ, csc Θ**	**cos Θ, sec Θ**	**tan Θ, cot Θ**
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative

Signs of the Six Trigonometric Functions – page 530	
Quadrant II (–, +) sin $\Theta > 0$, csc $\Theta > 0$ others negative	**Quadrant I** (+, +) All positive
Quadrant III (–, –) tan $\Theta > 0$, cot $\Theta > 0$ others negative	**Quadrant IV** (+, –) cos $\Theta > 0$, sec $\Theta > 0$ others negative

Section 6.5 – Unit Circle Approach; Properties of the Trigonometric Functions

Six Trigonometric Functions of *t* – page 538					
$\sin t = b$	$\cos t = a$	$\tan t = \frac{b}{a}$	$\csc t = \frac{1}{b}$	$\sec t = \frac{1}{a}$	$\cot t = \frac{a}{b}$

Domain and Range of the Six Trigonometric Functions – pages 541-542

Function	Symbol	Domain	Range
sine	$f(\Theta) = \sin \Theta$	All real numbers	All real numbers from –1 to 1, inclusive
cosine	$f(\Theta) = \cos \Theta$	All real numbers	All real numbers from –1 to 1, inclusive
tangent	$f(\Theta) = \tan \Theta$	All real numbers, except odd multiples of $\frac{\pi}{2}$ (90°)	All real numbers
cosecant	$f(\Theta) = \csc \Theta$	All real numbers, except integral multiples of π (180°)	All real numbers greater than or equal to 1; or less than or equal to –1
secant	$f(\Theta) = \sec \Theta$	All real numbers, except odd multiples of $\frac{\pi}{2}$ (90°)	All real numbers greater than or equal to 1; or less than or equal to –1
cotangent	$f(\Theta) = \cot \Theta$	All real numbers, except integral multiples of π (180°)	All real numbers

Periods of the Six Trigonometric Functions – page 543		
$\sin(\Theta + 2\pi) = \sin \Theta$	$\cos(\Theta + 2\pi) = \cos \Theta$	$\tan(\Theta + \pi) = \tan \Theta$
$\csc(\Theta + 2\pi) = \csc \Theta$	$\sec(\Theta + 2\pi) = \sec \Theta$	$\cot(\Theta + \pi) = \cot \Theta$

Even-Odd Properties of the Six Trigonometric Functions – page 544		
$\sin(-\Theta) = -\sin \Theta$	$\cos(-\Theta) = \cos \Theta$	$\tan(-\Theta) = -\tan \Theta$
$\csc(-\Theta) = -\csc \Theta$	$\sec(-\Theta) = \sec \Theta$	$\cot(-\Theta) = -\cot \Theta$

Section 6.6 – Graphs of the Sine and Cosine Functions

Properties of the Sine Function – page 549
1. The domain is the set of all real numbers.
2. The range consists of all real numbers from –1 to 1, inclusive.
3. The sine function is an odd function.
4. The sine function has a period of 2π.
5. The x-intercepts are…, $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi,\ldots$; the y-intercept is 0.
6. The maximum value is 1 and occurs at $x = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2},\ldots$; The minimum value is –1 and occurs at $x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2},\ldots$

Properties of the Cosine Function – page 551
1. The domain is the set of all real numbers.
2. The range consists of all real numbers from –1 to 1, inclusive.
3. The cosine function is an even function.
4. The cosine function has a period of 2π.
5. The x-intercepts are…, $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2},\ldots$; the y-intercept is 1.
6. The maximum value is 1 and occurs at $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi,\ldots$; The minimum value is –1 and occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi,\ldots$.

Section 6.7 – Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

Properties of the Tangent Function – page 566
1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$
2. The range is the set of all real numbers.
3. The tangent function is an odd function.
4. The tangent function has a period of π.
5. The x-intercepts are…, $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi,\ldots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2},\ldots$;

Section 6.8 – Phase Shift; Sinusoidal Curve Fitting

Amplitude, Period, and Phase Shift of Graphs – page 572

Amplitude = $|A|$ Period = $T = \frac{2\pi}{\omega}$ Phase Shift = $\frac{\Phi}{\omega}$

Phase Shift – page 571, 572

$y = A\sin(\omega x - \Phi) \quad = \quad y = A\sin\left[\omega\left(x - \frac{\Phi}{\omega}\right)\right]$

Graph begins at *Phase Shift*: $\omega x - \Phi = 0$ or $x = \frac{\Phi}{\omega}$

Graph ends at *Period* + *Phase Shift*: $\omega x - \Phi = 2\pi$ or $x = \frac{2\pi}{\omega} + \frac{\Phi}{\omega}$

Chapter 7 – Analytic Trigonometry

Section 7.1 – The Inverse Sine, Cosine, and Tangent Functions

Inverse Sine Function – page 593, 595

$y = \sin^{-1} x$ means $x = \sin y$ where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Restricted sin function: $f^{-1}(f(x)) = \sin^{-1}(\sin x) = x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Inverse sin function: $f(f^{-1}(x)) = \sin(\sin^{-1} x) = x$, where $-1 \le x \le 1$

Inverse Cosine Function – page 596, 598

$y = \cos^{-1} x$ means $x = \cos y$ where $-1 \le x \le 1$ and $0 \le y \le \pi$

Restricted cos function: $f^{-1}(f(x)) = \cos^{-1}(\cos x) = x$, where $0 \le x \le \pi$

Inverse cos function: $f(f^{-1}(x)) = \cos(\cos^{-1} x) = x$, where $-1 \le x \le 1$

Inverse Tangent Function – page 599, 600

$y = \tan^{-1} x$ means $x = \tan y$ where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Restricted tan function: $f^{-1}(f(x)) = \tan^{-1}(\tan x) = x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Inverse tan function: $f(f^{-1}(x)) = \tan(\tan^{-1} x) = x$, where $-\infty < x < \infty$

Section 7.2 – The Inverse Trigonometric Functions (Continued)

The Inverse Secant, Cosecant, and Cotangent Functions – page 605

$y = \sec^{-1} x$ means $x = \sec y$ where $|x| \geq 1$ and $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$

$y = \csc^{-1} x$ means $x = \csc y$ where $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$

$y = \cot^{-1} x$ means $x = \cot y$ where $-\infty < x < \infty$ and $0 < y < \pi$

Section 7.4 – Sum and Difference Formulas

Sum and Difference Formulas for Cosines – page 616

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Other Identities – page 617

$\cos\left(\frac{\pi}{2} - \Theta\right) = \sin\Theta$

$\sin\left(\frac{\pi}{2} - \Theta\right) = \cos\Theta$

Sum and Difference Formulas for Sines – page 619

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Sum and Difference Formulas for Tangents – page 621

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Section 7.5 – Double-angle and Half-angle Formulas

Double-Angle Formulas – page 626

$\sin(2\Theta) = 2\sin\Theta\cos\Theta$ $\quad$ $\cos(2\Theta) = \cos^2\Theta - \sin^2\Theta$

$\cos(2\Theta) = 1 - 2\sin^2\Theta$ $\quad$ $\cos(2\Theta) = 2\cos^2\Theta - 1$

Other Variations of the Double-Angle Formulas – page 627, 628

$\tan(2\Theta) = \dfrac{2\tan\Theta}{1-\tan^2\Theta}$ $\quad$ $\sin^2\Theta = \dfrac{1-\cos(2\Theta)}{2}$

$\cos^2\Theta = \dfrac{1+\cos(2\Theta)}{2}$ $\quad$ $\tan^2\Theta = \dfrac{1-\cos(2\Theta)}{1+\cos(2\Theta)}$

Half-Angle Formulas – page 630

$\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

$\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

$\tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

Half-Angle Formulas – page 630

$\sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$ $\quad$ $\cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$ $\quad$ $\tan^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{1+\cos\alpha}$

Half-Angle Formulas for $\tan\dfrac{\alpha}{2}$ – page 632

$\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha}$

Section 7.6 – Product-to-Sum and Sum-to-Product Formulas

Product-to-Sum Formulas – page 635

$\sin\alpha\sin\beta = \dfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$\cos\alpha\cos\beta = \dfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

$\sin\alpha\cos\beta = \dfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

Sum-to-Product Formulas – page 637

$\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

$\sin\alpha - \sin\beta = 2\sin\dfrac{\alpha-\beta}{2}\cos\dfrac{\alpha+\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

Chapter 8 – Applications of Trigonometric Functions

Section 8.2 – The Law of Sines

Law of Sines – page 670

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

$$\alpha + \beta + \gamma = 180°$$

Section 8.3 – The Law of Cosines

Law of Cosines to Find Third Side – page 681

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

Law of Cosines to Find Angle α – page 682

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Law of Cosines to Find Angle β – page 682

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$