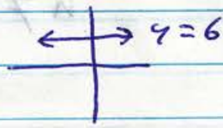
Limits

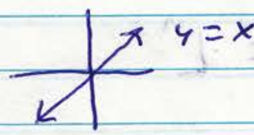
$$\lim_{x \to c} f(x) = L$$

Basic Limit Theorem pg 59

$$\lim_{x \to c} b = b$$ b and c real numbers



$$\lim_{x \to c} x = c$$ Identity function



$$\lim_{x \to c} x^n = c^n$$ $n > 0$

Properties of Limits pg 59

Let $\lim_{x \to c} f(x) = L$ $\quad \lim_{x \to c} g(x) = K$

Scalar Multiple

$$\lim_{x \to c} b \cdot f(x) = b \cdot \lim_{x \to c} f(x) = b \cdot L$$

Sum / Difference

$$\lim_{x \to c} \left[f(x) \pm g(x) \right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm K$$

Product Rule

$$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = L \cdot K$$

Quotient Rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K} \qquad K \neq 0$$

Power Rule

$$\lim_{x \to c} \left[f(x) \right]^n = L^n$$

Root Rule

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad , \; n > 0$$

(also negative numbers)
Limit is valid for all c when n is odd and valid for $c > 0$ when n is even.

Limit of Composite Functions

Let $\lim\limits_{x \to c} f(x) = L \qquad \lim\limits_{x \to c} g(x) = K$

$$\lim_{x \to c} f(g(x)) = \lim_{x \to c} f \circ g(x) = \lim_{x \to c} f(K)$$

Note: All methods on this page use direct substitution.

Limits of Trig Functions

$\lim_{x\to c} \sin x = \sin c$ $\lim_{x\to c} \csc x = \csc c$

$\lim_{x\to c} \cos x = \cos c$ $\lim_{x\to c} \sec x = \sec c$

$\lim_{x\to c} \tan x = \tan c$ $\lim_{x\to c} \cot x = \cot c$

for all real numbers c in the domain of each function

Squeeze Theorem

if f, g, h are functions where

$$h(x) \le f(x) \le g(x)$$

for all x in an open interval that contains c

and

$$\lim_{x\to c} h(x) = L$$

and

$$\lim_{x\to c} g(x) = L$$

then

$$\lim_{x\to c} f(x) = L$$

Two Special Trig Limits

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Definition of Continuity

Function f is continuous at c if the following 3 conditions exist:

1) $f(c)$ is defined
2) $\lim_{x \to c} f(x)$ must exist
3) $\lim_{x \to c} f(x) = f(c)$

Intermediate Value Theorem

To determine existence of zeros.

1) Ensure $f(x)$ is continuous on $[a, b]$. Polynomial functions are continuous.
2) Get function values of $f(a)$ and $f(b)$. Compare signs.
3) If signs opposite, there is at least one zero on $[a, b]$

Finding Vertical Asymptotes

1) Get values of x that make denominator $= 0$
2) Substitute founded values into numerator x. Then if num $= 0$, that x value substituted is not VA. If num $\neq 0$, that x value is VA.
3) List x values that are VAs; or say no VAs.

Note: if get $\frac{0}{0}$, can pursue further by taking limit, of discontinuity. If limit exists, then not VA. If limit is $-\infty$ or ∞, then VA.

Describe Unbounded Behavior - Infinite Limits

To get direction of VA & graph shape, check signs of $\frac{\text{num}}{\text{den}}$. Check limits from R & L by subbing numbers close to limit of discontinuity. Take limits

If:

$\frac{\text{"}-\text{"}}{\text{"}-\text{"}}$ then $+\infty$, $\frac{\text{"}+\text{"}}{\text{"}-\text{"}}$ or $\frac{\text{"}-\text{"}}{\text{"}+\text{"}}$ then $-\infty$

Continuity on Closed Interval [a, b]

1) Ensure endpoints a, b are defined.
   Is f(a) and f(b) defined?

2) Take Right / Left limits of a / b.

From Right	From Left
$\lim_{x \to a^+} f(x)$	$\lim_{x \to b^-} f(x)$

3) Do f(a/b) values equal their corresponding limit?
   Do:

$$\lim_{x \to a^+} f(x) = f(a) \quad \underline{\text{and}} \quad \lim_{x \to b^-} f(x) = f(b)$$

Point Discontinuity - Removable / Non-Removable

1) Factor f(x) and cross-out common factors.
   Find all discontinuities in den; set to 0.
2) Using each zero of denominator, take the limit on the factored out version of f(x).
   Substitute each zero for $\lim_{x \to ?}$

   If the result is a real number, the discontinuity is removable; got a limit; a hole.

   If result is $\frac{\text{num}}{0}$, then limit dne so discontinuity is non-removable.

   Note: shortcut is if factor in den is crossed out, then it is removable discontinuity.

3 Behaviors Associated with Non-Existence of Limit

1) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

2) $f(x)$ ↑ or ↓ without bounds as $x \to c$ from left, right, both

3) $f(x)$ oscillates as $x \to c$ from left, right, both

Continuity With 2 Terms in Piecewise Function and Inequality Conditionals

1) $f(c)$ is defined
2) Take right and left limits at c:

$$\lim_{x \to c^+} f(x) \qquad \lim_{x \to c^-} f(x)$$

will involve the appropriate term in definition; the term coupled to inequality conditional for c.

Ensure right and left limits are equal so we know limit exists.

3) $\lim_{x \to c} f(x) = f(c)$

Note: If either L & R limits are not equal or $\lim_{x \to c} f(x) \neq f(c)$, then nonremovable discontinuity.

Derivative of f(x)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Constant Function Rule

$$\frac{d}{dx}[c] = 0$$

Power Rule

$f(x) = x^n$, then $f'(x) = n \cdot x^{n-1}$

Constant Multiple Rule

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

Sum and Difference Rules

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Differentiating sin and cos

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Four Steps to Find Derivative

Step 1 Find $f(x+\Delta x)$

Step 2 Find $f(x+\Delta x) - f(x)$

Step 3 Find $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Step 4 Find $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Then $f'(a)$ as needed

Find Equation of a Tangent Line

Step 1 - Find a point. Already given one coordinate so plug x into f(x) to get y coordinate.

Step 2 - Find slope.

Step 2a - Get derivative

Step 2b - Get slope by substituting given x into derivative.

Step 3 - Use Point-Slope formula and plug in point and slope.

$$y - y_1 = m(x - x_1)$$

Rewrite as Slope-Intercept → $y = mx + b$

Find Point(s) of Horizontal Tangent

Step 1 - Take derivative

Step 2 - Set derivative to zero and solve for X

Step 3 - Get y values for each X by substituting into original function.

Displacement / Position / Distance Problems

$$r = \frac{d}{t} \Rightarrow \text{rate} = \frac{\text{distance}}{\text{time}}$$

$$\text{Average Velocity} = \frac{\Delta d}{\Delta t}$$

Displacement Equation of Free-Falling Object on Ground

$$s(t) = \frac{1}{2} g t^2$$

$$g = -32 \text{ ft/sec}^2 \text{ or } g = -9.8 \text{ m/sec}^2$$

Displacement Function of Free-Falling Object with Initial Position

$$s(t) = \frac{1}{2} g t^2 + s_0$$

← initial position

Displacement Function of Free-Falling Object with Initial Position and Initial Velocity

$$s(t) = \frac{1}{2} g t^2 + v_0 t + s_0$$

↑ initial velocity

Instantaneous Velocity

$s'(t) = v(t)$

Questions Involving Position & Velocity

- write function that describes velocity of ball at any time

indicates generic velocity function

$s_0, \; s'(t) = v(t)$ Take derivative

- what is velocity after 3 seconds?

$v(3)$

- How many seconds for ball to reach ground?

Set position function to zero (ground) and solve for t.

$s(t) = 0$ Exclude negative seconds

- what is velocity of ball at impact? At "x" seconds?

Occurs when $s(t) = 0$ found above.

Then plug seconds found into $v(t)$.

- what is velocity after ball falls 108 feet?

Step 1 - Find height of ball from ground since ground is zero.
Initial Position - Feet Fallen = Feet from Ground

Step 2 - Set position function equal to ball "feet from ground"
Ex: $16t^2 - 22t + 220 = 112$

Step 3 - Solve for time (t). Exclude negative seconds.

Step 4 - Plug seconds found into velocity function.
Ex: $v(2) = -32(2) - 22 = -86$ feet/sec

Product Rule

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

or

$$gf' + fg'$$

3 functions: $f'gh + fg'h + fgh'$

Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$$

or

$$\frac{gf' - fg'}{g^2}$$

Derivatives of 4 More Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$ $\qquad$ $\csc^2 x - 1 = \cot^2 x$

$\tan^2 x + 1 = \sec^2 x$

Chain Rule

To differentiate composite functions.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{derivative of outside function}} \cdot \underbrace{g'(x)}_{\text{derivative of inside function}}$$

Another way to define Chain Rule:

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{\text{outer}} \cdot \underbrace{\frac{du}{dx}}_{\text{inner}}$$

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General Power Rule

Special case of the Chain Rule.

$$\frac{d}{dx}\left[u^n\right] = n u^{n-1} u'$$

Another way to define General Power Rule:

$$\frac{dy}{dx} = n\left[u(x)\right]^{n-1} \frac{du}{dx}$$

← inner

Trigonometric Functions and Chain Rule

$\frac{d}{dx}\left[\sin u\right] = (\cos u)u'$ $\frac{d}{dx}\left[\cos u\right] = -(\sin u)u'$

$\frac{d}{dx}\left[\tan u\right] = (\sec^2 u)u'$ $\frac{d}{dx}\left[\cot u\right] = -(\csc^2 u)u'$

$\frac{d}{dx}\left[\sec u\right] = (\sec u \tan u)u'$ $\frac{d}{dx}\left[\csc u\right] = -(\csc u \cot u)u'$

Solving Implicit Differentiation

1) Differentiate both sides wrt "x" $\frac{d}{dx}[\ldots]$
2) Get $\frac{dy}{dx}$ terms on one side
3) Factor $\frac{dy}{dx}$ out of terms
4) Divide both sides by "coeficcents" of $\frac{dy}{dx}$ to solve for $\frac{dy}{dx}$

Related Rates

$\frac{dv}{dt}$ = instantaneous rate of change of volume wrt time

$\frac{dr}{dt}$ = instantaneous rate of change of radius wrt time

Steps in Solving Related Rates

Step 1 - Draw picture
- Identify variables, known & unknown
- Identify rates, known & unknown
- Identify goal, the unknown rate

Step 2 - Write equation that relates all variables.

Step 3 - Differentiate equation wrt time, $\frac{d}{dt}$ to get related rate.

Step 4 - Substitute known values & solve.

Test for Symmetry

wrt y-axis
- Substitute $-x$ for x into equation
- Simplify
- If same as original equation, have symmetry.

wrt x-axis
- Substitute $-y$ for y into equation
- Simplify
- If same as original equation, have symmetry.

wrt origin
- Substitute $-x$ for x and $-y$ for y into equation.
- Simplify
- If same as original equation, have symmetry.

Extreme Value Theorem

If a function is continuous on a closed interval $[a, b]$, then the function has both a max and min on the interval.

Finding Critical Numbers

1) $f'(x)$
2) $f'(x) = 0$ } Possible relative extrema
3) $f'(x) = $ dne?

Find Absolute Extrema

1) Find critical numbers (see above)
2) Evaluate f at each critical # in (a, b)
3) Evaluate f at each endpoint of $[a, b]$
4) Compare values of 2 and 3 above. Least is absolute min & most is absolute max.

Rolle's Theorem

Guarantees existence of point between $x=a$ and $x=b$ where tangent line is horizontal, $f'(c) = 0$

Mean Value Theorem

Guarantees one point between $x=a$ and $x=b$ where slope of tangent is equal (parallel) to slope of secant line

$$\text{Slope of secant} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Slope of tangent at } x = c = f'(c)$$

Theorem 3.5

- If $f'(x) > 0$ (positive), f is increasing
- If $f'(x) < 0$ (negative), f is decreasing
- If $f'(x) = 0$, f is constant

First Derivative Test

- If $f'(x)$ changes $-$ to $+$, $f(c)$ is relative min
- If $f'(x)$ changes $+$ to $-$, $f(c)$ is relative max
- If $f'(x)$ does not change signs, neither rel min nor max

Find Relative Extrema Using FDT

Step 1 - Find critical numbers
Step 2 - Intervals table
Step 3 - FDT conclusions
Step 4 - Graph; x and y int, f(critical #'s)

Concavity

- Increasing slope ⇒ concave up
- Decreasing slope ⇒ concave down

Theorem 3.7

- If $f''(x) > 0$ (positive), f is concave up
- If $f''(x) < 0$ (negative), f is concave down

Points of Inflection

- Occur when $f''(x) = 0$ or $f''(x) = \text{dne}$
- These are PPOI, not guaranteed POI
(sign change of $f''(x)$ in table means POI)

Second Derivative Test

- If $f''(c) > 0$ (positive), f has relative min at c ∪
- If $f''(c) < 0$ (negative), f has relative max at c ∩
- If $f''(c) = 0$, test fails so must use FDT to determine relative max or min

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Find Relative Extrema Using SDT

Step 1 - Find critical values
Step 2 - Use SDT to find relative extrema by substituting critical #'s into $f''(x)$ to get sign
Step 3 - Intervals table for concavity & POI
 a) Set $f''(x) = 0$ } PPOI's
 b) $f''(x) = $ dne? }
 c) Concavity table
Step 4 - Got POI's? If - to +, or + to - of $f''(x)$ sign in table
Step 5 - Graph; set up table of x values & corresponding $f(x)$ coordinates.

	max	min	POI	x-int	y-int
x					
$f(x)$					

Limits at Infinity

$$\lim_{x \to \pm\infty} \frac{c}{x^r} = 0$$

1) When degree of numerator = degree of denominator, then HA is ratio of leading coefficients.
2) When degree of num < degree of den, HA is $Y = 0$
3) When degree of num > degree of den, HA is ∞, limit dne.

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0 \text{ by Squeeze Theorem}$$

Vertical Asymptote, as $x \to c$, $f(x) \to \pm\infty$

$\lim_{x \to c} f(x) = \pm\infty$ $x = c$ is V.A.

Horizontal Asymptote, as $x \to \pm\infty$ $f(x) \to L$

$\lim_{x \to \pm\infty} f(x) = L$ $\quad y = L$ is H.A.

Find & Plot VA & HA

Step 1 - Find VA

a) Set den $= 0$; check num $\neq 0$ with found values of x

Step 2 - Use sign line to determine behavior L & R of VA.

a) Choose number to left of VA & take limit.

b) Choose number to right of VA & take limit.

If sign is negative $\left(\frac{+}{-}, \frac{-}{+}\right)$, function is decreasing to that side of VA.

If sign is positive $\left(\frac{+}{+}\right)$, function is increasing on that side of VA.

Step 3 - Find HA

num = den $\Rightarrow$ ratio leading coefficients

num > den $\Rightarrow \infty$, no H.A.

num < den $\Rightarrow$ 0, $y = 0$

Step 4 - Determine end behavior, above or below HA?

Positive x-axis infinity

a) Take limit of large positive number (1000) and compare the result with value of HA. Is limit number below or above HA value for $x \to +\infty$?

b) Negative x-axis infinity - Take limit of large negative number (-1000) and compare result with value of HA. Is limit number below or above HA value for $x \to -\infty$?

Step 5 - Graph

Curve Sketching - Analyze & Sketch

Step 0 - Find $f'(x)$ and $f''(x)$

Step 1 - Use $f(x)$ to find:

a) x-intercept - set num = 0 get (x, y)

b) y-intercept - set $x = 0$ get (x, y)

c) Symmetry

1) y-axis: set $(-x)$

2) x-axis: set $(-f(x))$

3) origin: set $(-f(x))$ and $(-x)$

d) VA - set den = 0 - ensure x values do not make num = 0

e) HA - compare num & den degree

Step 2 -

a) Critical numbers - use $f'(x)$

1) Set num = 0

2) $f'(x)$ = dne?

b) PPOI's - use $f''(x)$

1) Set num = 0

2) $f''(x)$ = dne?

Step 3 - Partition obtained x-values into intervals.

x-int VA Critical #s PPOIs

Ex: $x=-1$ $x=0$ $x=1$

Step 4 - Use intervals table to calculate sign of y, y', y''

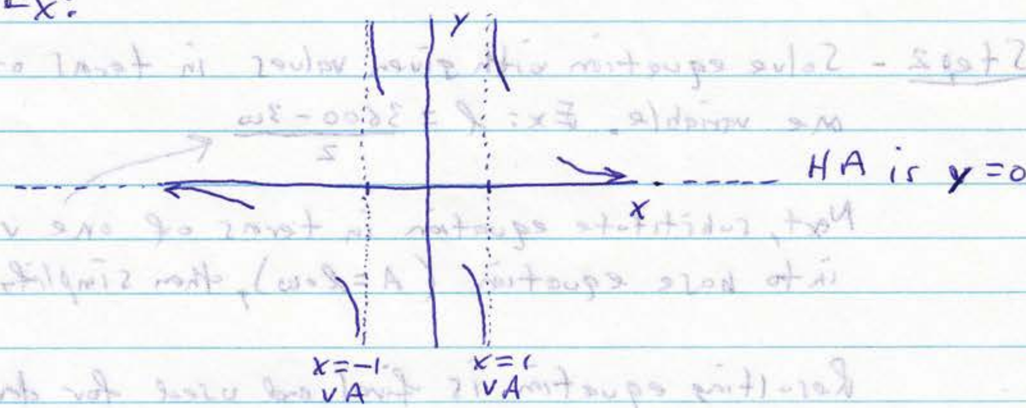
	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
x	-10	$-.5$	$.5$	10
$f(x)$ sign	$\frac{-30}{99} < 0$ HA	$\frac{-}{-} = +$	$\frac{+}{-} = -$	$\frac{+30}{99} > 0$ HA
$f'(x)$ sign	$\frac{-}{+} = -$	$\frac{-}{+} = -$	$\frac{-}{+} = -$	$\frac{-}{+} = -$
$f''(x)$ sign	$\frac{-}{+} = -$	$\frac{-}{-} = +$	$\frac{+}{-} = -$	$\frac{+}{+} = +$
Conclusion	Decreasing Concave down	Decreasing Concave up	Decreasing Concave down	Decreasing Concave up

from below HA

from above HA

Step 5 - Use intervals table results to determine extrema, POI, VA end behavior, HA end behavior.

Ex:



a) Use f(x) sign to determine HA and behavior.
Left from below. Right from above.

b) Use limits to determine VA end behavior

Ex: $x=-1$

$\lim_{x\to -1^-} \frac{3(-2)}{(-2)^2-1} = \frac{-6}{3} = -2$, negative so decreases to left

$\lim_{x\to -1^+} \frac{3(-.5)}{(-.5)^2-1} = \frac{-1.5}{-.75} = +2$, positive so increasing to right

$x=+1$

$\lim_{x\to +1^-} \frac{3(+.5)}{(+.5)^2-1} = \frac{1.5}{-.75} = -2$, negative so decreasing to left

$\lim_{x\to +1^+} \frac{3(2)}{(2)^2-1} = \frac{6}{3} = +2$, positive so increasing to right

c) Use $f'(x)$ sign to determine relative extrema.

d) Use $f''(x)$ sign to determine POI's.

Step 6 - Graph

a) Create table of (x, y) coordinate values.

Ex:

	x-int	y-int	Critical #	Critical #	POI	POI
x						
f(x)						

b) Use all information to graph. Include and label points.

(10)

Optimization

Ex: $A = l \cdot w$

Step 1 - List known items. What is the goal? List base equation.

Use given values to build an equation. Ex: $3600 = 3w + 2l$

Step 2 - Solve equation with given values in terms of one variable. Ex: $l = \frac{3600 - 3w}{2}$

Next, substitute equation in terms of one variable into base equation ($A = l \cdot w$), then simplify.

Resulting equation is final and used for derivative.

Ex: $A(w) = \left(\frac{3600 - 3w}{2}\right) \cdot w$

Final equation $\longrightarrow$ $A(w) = 1800w - \frac{3}{2}w^2$

Step 3 - Find critical numbers by differentiating, then set $= 0$.

a) $A'(w) = 1800 - 3w$

b) Set $A'(w) = 0 \Rightarrow 1800 - 3w = 0 \Rightarrow 1800 = 3w \Rightarrow w = 600$

Step 4 - Use FDT or SDT to confirm relative max or min.

a) FDT

	$(0, 600)$	$(600, 3600)$
w	400	1000
$A'(w)$ sign	$+$	$-$

Increasing $(+)$ to decreasing $(-)$ so relative max.

b) SDT

$A''(w) = -3$, negative constant so always concave down $\cap$ so rel max

Step 5 - Conclusions, answer the question.

In above example,

Found $w = 600$ — Substitute critical number found (confirmed rel max) into equation derived from given values into terms of one variable $\left(l = \frac{3600 - 3w}{2}\right)$

$$l = \frac{3600 - 3(600)}{2} = 900$$

Substitute both values found into base equation $A = l \cdot w$

$A = 900 \cdot 600 = 540{,}000\ yd^2$

Geometry Formulas for Optimization

Area $= l \cdot w$

Volume $= l \cdot w \cdot h$

or $x^2 \cdot h$ for square box

Numbers whose product is...

$P = x \cdot y$

Distance Formula

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Newton's Method

To find zeros of function (x-int.)

a) Guess first number so that it is between x numbers that cause $f(x)$ to have opposite signs. Intermediate Value Theorem guarantees...

Ex: $f(x) = x^2 - 11$ $\quad x_1 = 3$ guess

$\left.\begin{array}{l} f(2) = -7 \\ f(4) = +5 \end{array}\right\}$ sign changes so $x_1 = 3$ is a good guess $\left(\begin{array}{l}\text{between} \\ 2 \text{ and } 4\end{array}\right)$

b) Calculate x_2, x_3 ... based on $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

Include all digits displayed on calculator for each calculation.

Linear Approximation, AKA Tangent Line Approximation

$$\boxed{y = f(c) + f'(c)(x - c)}$$

Only used for values near c. Otherwise need to create new linear approximation near the new c.

Differential of y

$$\boxed{dy = f'(x)\,dx}$$

Approximates value of function

Differential Linear Approximation, AKA Tangent Line Approximation

when dx is very small.

$$\boxed{f(x + dx) \approx f(x) + f'(x) \cdot dx}$$

Provides function values for any c for that specific function only.

Propagated Error, AKA Error in Output

Use $dy \approx f'(x)\,dx$, if dx is small

Ex: $x = 12$ in. $\quad dx = \pm \frac{1}{64}$ inch

Base equation → $A = x^2$

$$dy \approx f'(x)\,dx$$
$$dA = A'\,dx$$
$$dA = 2x\,dx$$
$$dA = 2(12)\left(\pm \tfrac{1}{64}\right)$$
$$dA = \pm \tfrac{3}{8}\text{ in}^2 \quad \text{Propagated error in Area}$$

Then, approximate percentage of change in error:

$dA = \pm \frac{3}{8}\text{ in}^2$, $\quad A = 12^2 = 144\text{ in}^2$

So,

$$\frac{dA}{A} = \frac{\pm \frac{3}{8}}{144} = \pm 0.0026 \Rightarrow 0.26\% \text{ error} \quad \left(\begin{matrix}\text{relative}\\ \text{error}\end{matrix}\right)$$

Chapter 4 - Integration

Section 4.1

Basic Integration Rules

a) $\int k\,dx = kx + C$

b) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$

c) $\int \sin x\,dx = -\cos x + C$

d) $\int \cos x\,dx = \sin x + C$

e) $\int \sec^2 x\,dx = \tan x + C$

f) $\int \sec x \tan x\,dx = \sec x + C$

g) $\int \csc^2 x\,dx = -\cot x + C$

h) $\int \csc x \cot x\,dx = -\csc x + C$

i) $\int k f(x)\,dx = k\int f(x)\,dx$

j) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Indefinite Integral

$$\int f(x)\,dx = F(x) + C$$

Find Particular Solution for Integration

Step 1 - Find general solution: Integrate both sides.

End up with $y = x + C$

Step 2 - Find particular solution. Substitute initial condition given for x and y, into general solution.

End up with $C =$ number

Step 3 - Substitute C into general solution to get particular solution.

Find Particular Solution for $f''(x)$

Step 1 - Find general solution for $f'(x)$

End up with $f'(x) = x + C$

Step 2 - Find particular solution for $f'(x)$. Substitute initial condition for $f'(x)$.

End up with $C =$ number

Step 3 - Substitute C into $f'(x)$ general solution.

Step 4 - Find general solution for $f(x)$.

End up with $f(x) = x + C$

Step 5 - Find particular solution for $f(x)$. Substitute initial condition for $f(x)$.

End up with $C =$ number

Step 6 - Substitute C into $f(x)$ general solution.

Vertical Motion and the Integral

Recall $s(t)$ position function

$\frac{ds}{dt} = v(t)$ velocity function

$\frac{dv}{dt} = a(t)$ acceleration function

Integrating velocity $\Rightarrow$ position $s(t) = \int v(t) \cdot dt$

Integrating acceleration $\Rightarrow$ velocity $v(t) = \int a(t) \cdot dt$

Constants of $a(t)$ for gravity $-9.8\ m/sec^2$ or $-32\ ft/sec^2$

(12)

Example of Vertical Motion and Integral

1) Start by finding general then particular solution for velocity using $v(t) = \int a(t)\,dt \Rightarrow v(t) = -9.8t + C \Rightarrow$
   $v(t) = -9.8t + \underset{\text{initial velocity.}}{49}$
2) Find general then particular solution for position using $s(t) = \int v(t)\,dt \Rightarrow s(t) = -4.9t^2 + 49t + C \Rightarrow$
   $s(t) = -4.9t^2 + 49t + \underset{\text{initial position (on earth).}}{0}$
   Same format as $s(t) = \frac{1}{2}g(t)^2 + V_0t + S_0$
3) Answer questions:
   a) Find max height of projectile.
      i) Find critical #'s. Set velocity function $= 0$ and solve for t. Setting $v(t) = 0$ since that is max height reached.
      ii) Verify relative max using SDT: $s''(t) = a(t) = -9.8$ ∩
      iii) Substitute value of t into position function to get height in m/ft
   b) When does projectile hit ground?
      i) Set $s(t) = 0$ since that is ground. If get 2 answers for t one is 0 which is right before takeoff.
   c) What is impact velocity?
      i) Know from above impact is at time $\Rightarrow t = 10$ sec.
      ii) Substitute $t = 10$ into velocity function
      iii) Answer is in $-$m/sec (or $-$ft/sec).

Section 4.2 - Area

Sum of n terms: $\sum_{i=1}^{n} a_i = a_1 + a_2 \cdots + a_n$

Summation Properties

$$\sum_{i=1}^{n} c \cdot a_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i$$, m begins at value other than 1

Area for Right-Handed Values

Area Under a Curve

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f(a + i \cdot \Delta x)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$$

$$\text{width} = \frac{b-a}{n} = \Delta x$$

n = # of rectangles

Use Right Hand Values to Approximate Area Under Curve

Step 1 - Find width of each rectangle: $\Delta x = \frac{b-a}{n}$

Step 2 - Find height of each rectangle: $f(a + i \cdot \Delta x)$
 1) Substitute a and Δx into $\longrightarrow$ and solve

Step 3 - Find area: $f(a + i \cdot \Delta x) \cdot \Delta x$
 1) Multiply height and width.

Step 4 - Set up summation, use summation formulas and properties to sum up area.
 - Take constant to left for each term to leave i variable or the constant 1 in Σ.
 - Use properties and formulas. Use formula based on degree of i.
 - Simplify

Step 5 - Take limit of each term.
 - Answer of area is in square units.

General Formula for Area Under Curve

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x$$

left-hand right hand

$$x_{i-1} \le c_i \le x_i$$

More general than left- or right-hand values.

Riemann Sum

$$\lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

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Definite Integral

$$\int_a^b f(x) \cdot dx$$

Properties of Definite Integrals

① $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$

② $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Special Definite Integrals

① If f is defined at $x = a$,

$$\int_a^a f(x)\,dx = 0$$

② If f is integrable on $[a, b]$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Section 4.4 - Fundamental Theorem of Calculus

Antiderivative Form of Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Evaluating Definite Integral:
1) Find antiderivative
2) Evaluate at limits
3) Subtract

Mean Value Theorem for Integrals

$$\int_a^b f(x)\,dx = f(c) \cdot (b-a)$$

Average Value of a Function

$$\frac{1}{b-a} \cdot \int_a^b f(x)\,dx$$

Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Section 4.5 - Integration by Substitution

Integration of Composite Function

$$\int f(u) \cdot u'(x)\,dx = F(u(x)) + C$$

Steps to Using u-Substitution for Indefinite Integral

Step 1 - Appropriate substitution for u. Usually term(s) in parenthesis. It can be:

a) Alegraic expression raised to a power

b) Angle of trig function

c) Trig function raised to a power

Step 2 - Find $\frac{du}{dx}$

Step 3 - Substitute u-expressions for x-expressions

Step 4 - Integrate wrt u

Step 5 - Back-substitute x-expressions for u-expressions.

Steps for u-Substitution for Definite Integral

Version 1 - Do all 5 steps above then solve using integration limits (since now based on x limit values).

Version 2 - Step 3 is different than above. Find u-values for integration limits by converting x to u values.

from step 1 → Ex: $u = x^2 + 1$

When $x=0$	When $x=1$
$u = 0^2 + 1$	$u = 1^2 + 1$
$u = 1$	$u = 2$

cont.

Now, $\int_1^2 u^3 \cdot \frac{1}{2} du$

Then Step 4 is integration, $\frac{1}{8} u^4 \Big|_1^2 \Rightarrow \frac{1}{8}(2^4 - 1^4) = \frac{15}{8}$

No Step 5 in version 2; no back substitution; got answer using u limits.

Finding Area Bounded by x-axis, f(x), on [a, b]

Step 1 - Find limits of integration:

a) Get x-intercepts; set $f(x) = 0$

b) Use x-intercepts to partition [a, b] into "+" or "-" subintervals

c) Substitute f(c) in each subinterval to see if f(x) is "+" or "-"

Endpoints of subintervals become limits of integration.

Step 2 - Integrate each subinterval and add.

$$\int_a^b f(x)\,dx \oplus \int_b^c f(x)\,dx$$

Step 3 - Add absolute values of each definite integral.

Finding Area Between 2 Curves on [a, b]

Step 1 - Find limits of integration:

a) Get points of intersection between curves. These points (x values) will be the limits.

b) Set 2 functions equal to each other

$f(x) = g(x)$

c) Solve for x('s). Will get 2 limits exactly (for this case)

Step 2 - Determine which function is greater (above) on each subinterval.

a) Pick x value for f(x) and g(x) between (a, b). Resulting y value determines which function is greater.

Step 3 - Set up integral so that greater function goes first.

$$\int_a^b \big(f(x) - g(x)\big)\,dx$$

Step 4 - Subtract 2 functions and simplify

Step 5 - Integrate, substitute limits, solve.

Find Area Between 2 Curves that Intersect in More than 2 Points

Step 1 - Find limits of integration:

a) Get points of intersection between curves. These x values will be the limits.

b) Set 2 functions equal to each other
$f(x) = g(x)$

c) Solve for x('s). Will get > 2 limits (points of intersection).

Step 2 - For each interval $[a, b]$, $[b, c]$ based on limits found above:

a) Find which function is above or below.

b) Pick x value for each subinterval and substitute into $f(x)$ and $g(x)$.

c) Resulting y value determines which function is greater for each subinterval.

Ex: $[a, b]$ $\qquad$ $[b, c]$

$[a, b]$	$[b, c]$
let $x = -1$	let $x = 1$
$y_1 = 6$	$y_1 = -8$
$y_2 = -3$	$y_2 = 1$
$y_1 > y_2$	$y_2 > y_1$

Step 3 - Set up 2 integrals:

$$\int_a^b (y_1 - y_2)\,dx + \int_b^c (y_2 - y_1)\,dx$$

Greater function goes first for each integral

Step 4 - Subtract 2 functions within the 2 integrals

Step 5 - Integrate, substitute limits, simplify each of the 2 integrals.

Step 6 - Add the 2 integral results to get square units.

Volume of Solid of Revolution: Disk Method Rotating About x-axis

$$V = \pi \int_a^b R(x)^2\,dx$$

- Limits are x values
- Radii are y values

Step 1 - Draw region. Determine method (disk or washer).

a) Find limits: (Problem may state; bounded by 1st quadrant means $x=0$)

1) y-intercept $\Rightarrow$ set $x=0$ $\Rightarrow$ Therefore $x=0$
2) x-intercept $\Rightarrow$ set $y=0$ $\Rightarrow$ solve for x
3) Use x-values from 2 steps above for limits

Step 2 - Revolve region about x-axis.

a) Find radii, which will be y values. So $y=-2x+2 \Rightarrow$ $R(x) = -2x+2$, for example.

Step 3 - Set up integral.

Ex:

$$V = \pi \int_0^1 [\underbrace{-2x+2}_{R(x)}]^2\,dx$$

a) let $u=-2x+2 \Rightarrow du = -2dx \Rightarrow dx = \frac{du}{-2}$

b) Integrate.

$$V = \pi \int u^2 \cdot -\tfrac{1}{2}\,du$$

$\vdots$

answer in cubic units

Volume of Solid of Revolution: Disk Method Rotating About y-axis

$V = \pi \int_a^b R(y)^2 \, dy$ - Limits are y values
- Radii are x values

Step 1 - Draw region. Determine method (disk or washer).
a) Find limits:
1) y-intercept ⇒ set $x=0$ ⇒ solve for y
2) x-intercept ⇒ set $y=0$ ⇒ Therefore $y=0$
3) Use y-values from 2 steps above for limits

Step 2 - Revolve region about y-axis.
a) Find radii, which will be x values.
1) Radii are in terms of $R(y) = x$ so need to solve function for x.
Ex: $y = -2x + 2 \Rightarrow x = \dfrac{2-y}{2} \Rightarrow -\frac{1}{2}y + 1 = R(y)$

Step 3 - Set up integral.
Ex:
$$V = \pi \int_0^2 \left(-\tfrac{1}{2}y + 1\right)^2 dy$$

a) Integrate wrt y, $\frac{du}{dy}$

b) Let $u = -\frac{1}{2}y + 1 \Rightarrow du = -\frac{1}{2}\,dy \Rightarrow dy = -2\,du$

c) Integrate.
$$V = \pi \int u^2 \cdot (-2)\, du$$
⋮
answer in cubic units

(21) (1c) Use washer method if there is a gap between axis of rotation and region.

Volume of Solid of Revolution: Washer Method Rotating About x-axis

$V = \pi\int_a^b \left[R(x)^2 - r(x)^2\right] dx$ - Limits are x values
- Radii are y values

Step 1 - Draw region. Determine method (disk or washer).

a) Find limits.

1) Use y-intercept (set $x=0$) and x-intercept (set $y=0$).
2) May be given one x value (limit) in problem; first quadrant area means $x=0$.
3) Another x value (limit) will be point of intersection.
   i) Set 2 function equal to each other and solve for x.
4) Use x values from steps above for limits.

Step 2 - Revolve region about x-axis.

a) Find radii, which will be y values.

1) Outer radii $R(x)$ is the function further from axis of revolution.
2) Inner radii $r(x)$ is the function closer to axis of revolution.

Step 3 - Set up integral and integrate.

Ex:

$$V = \pi\int_0^4 \left[2^2 - (\sqrt{x})^2\right] dx$$

$R(x)^2$ goes first $\qquad$ $r(x)^2$ goes second

a) Expand out each function from power of 2.
b) Subtract functions.
c) Simplify.
d) Integrate.
e) Substitute limits.
f) Answer in cubic units.

Volume of Solid of Revolution: Washer Method Rotating About y-axis

$$V = \pi \int_a^b \left(R(y)^2 - r(y)^2 \right) dy$$

- Limits are y values
- Radii are x values

Step 1 - Draw region. Determine method (disk or washer).

a) Find limits.

1) Use y-intercept (set $x=0$) and x-intercept (set $y=0$).
2) May be given one y value (limit) in problem; first quadrant area means $y=0$.
3) Another y value (limit) will be point of intersection.
   i) Solve equations for x.
      Ex: $y = x^2 \Rightarrow \sqrt{y} = \sqrt{x^2} \Rightarrow x = \sqrt{y}$
   ii) Set 2 functions equal to each other and solve for y.
4) Use y values from steps above for limits.

Step 2 - Revolve region about y-axis.

a) Find radii, which will be x values.

1) Outer radii $R(y)$ is the function further from axis of revolution.
2) Inner radii $r(y)$ is the function closer to axis of revolution.

Step 3 - Set up integral and integrate.

Ex:

$$V = \pi \int_0^4 \left[(2)^2 - (\sqrt{y})^2 \right] dy$$

$R(y)^2$ goes first $\qquad$ $r(y)^2$ goes second

a) Expand out each function from power of 2.
b) Subtract functions.
c) Simplify.
d) Integrate
e) Substitute limits.
f) Answer in cubic units.