Chapter 12 Study Notes

Theorem 12.2 Properties of the Derivative

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be a scalar.

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$ — Product Rule
4. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$ — Same as Product Rule but using Dot Product
5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$ — Order of the Cross-Product matters

Remark

Note that Property 5 applies only to three-dimensional vector-valued functions because the cross product is not defined for two-dimensional vectors.

6. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$ — Composite function, use Chain Rule
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Example 4 Using Properties of the Derivative

For $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}$ and $\mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$, find each derivative.

a. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

b. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)]$

Solution

a. Because $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$, you have

$$\begin{aligned}\tfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\tfrac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) + \left(-\tfrac{1}{t^2}\mathbf{i} + \tfrac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \tfrac{1}{t} \\ &= 3 + \tfrac{1}{t}.\end{aligned}$$

b. Because $\mathbf{u}'(t) = 2t\mathbf{i} - \mathbf{j}$ and $\mathbf{u}''(t) = 2\mathbf{i}$, you have

$$\begin{aligned}\tfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}\mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}.\end{aligned}$$

Definitions of Velocity and Acceleration

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

Two Dimensions

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Magnitude of Velocity

For motion along a space curve, the definitions are similar. That is, for

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

you have the following.

Three Dimensions

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0 \qquad \text{Position vector}$$

$$= -\tfrac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 \cos\theta\mathbf{i} + t\mathbf{v}_0 \sin\theta\mathbf{j} + h\mathbf{j}$$

$$= (\mathbf{v}_0 \cos\theta)\,t\mathbf{i} + \left[h + (\mathbf{v}_0 \sin\theta)\,t - \tfrac{1}{2}gt^2\right]\mathbf{j}.$$

Theorem 12.3 Position Vector for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos\theta)\,t\mathbf{i} + \left[h + (v_0 \sin\theta)\,t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the acceleration due to gravity.

Summary of Velocity, Acceleration, and Curvature

Unless noted otherwise, let C be a curve (in the plane or in space) given by the position vector

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{Curve in the plane}$$

Position Vectors

or

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad \text{Curve in space}$$

where x, y, and z are twice-differentiable functions of t.

Velocity vector, speed, and acceleration vector

$$\mathbf{v}(t) = \mathbf{r}'(t) \qquad \text{Velocity vector}$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt} \qquad \text{Speed}$$

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{r}''(t) \qquad \text{Acceleration vector}\\ &= a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)\\ &= \frac{d^2s}{dt^2}\mathbf{T}(t) + K\left(\frac{ds}{dt}\right)^2\mathbf{N}(t) \qquad K \text{ is curvature and } \frac{ds}{dt} \text{ is speed.}\end{aligned}$$

Unit tangent vector and principal unit normal vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \text{Unit tangent vector}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \qquad \text{Principal unit normal vector}$$

Velocity Vector, aka Tangent Vector

Summary of Velocity, Acceleration, and Curvature
(Continued)

Components of acceleration

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{d^2 s}{dt^2}$$ Tangential component of acceleration

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N}$$ Normal component of acceleration

$$= \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$= \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

$$= K\left(\frac{ds}{dt}\right)^2$$ K is curvature and $\frac{ds}{dt}$ is speed.

Formulas for curvature in the plane

$$K = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$ C given by $y = f(x)$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2+(y')^2\right]^{3/2}}$$ C given by $x = x(t)$, $y = y(t)$

Formulas for curvature in the plane or in space

$$K = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$$ s is arc length parameter.

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$ t is general parameter.

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$

Cross product formulas apply only to curves in space.

Arc Length of a Space Curve:

$$s = \int_a^b \|\boldsymbol{r}'(t)\| dt$$